Imperial College London

Module 3 Class

Aidan Hogg - 31 October 2019
ELEC96010 (EE3-07): Digital Signal Processing
Department of Electrical and Electronic Engineering

Method:

1: Conceptual question posed - students think quietly on their own and report initial answers on Mentimeter **(3 mins)**
2: Students discuss their answers in small groups **(2 mins)**
3: Explanation/discussion of correct answer **(3 mins)**

The normal arithmetic rules for multiplication are:

Identity: $x[n] * \delta[n] = x[n]$

Commutative: $x[n] * v[n] = v[n] * x[n]$

Associative: $x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$

Distributive: $x[n] * (\alpha v[n] + \beta w[n]) = (x[n] * \alpha v[n]) + (x[n] * \beta w[n])$

What normal arithmetic rules does Convolution obey?

A: Identity and commutative

B: Identity, commutative and associative

C: All of them

ANSWER

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What normal arithmetic rules does Convolution obey?

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Consider the finite length signals

$$x[0], \cdots, x[N-1]$$
$$y[0], \cdots, y[M-1] \qquad \text{where } M < N$$

When

$$v[n] = x[n] * y[n]$$
$$w[n] = x[n] \circledast_N y[n]$$

State the values of n for which $v[n] = w[n]$?

A: $0 \leq n \leq N-1$

B: $M-1 \leq n \leq N-1$

C: $N-M \leq n \leq N-1$

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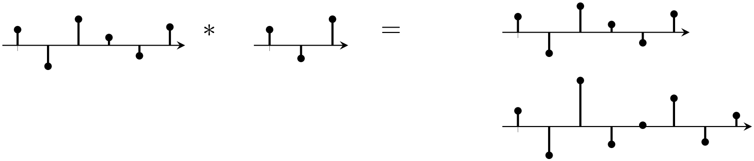
Linear Convolution

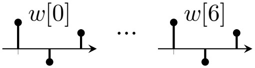


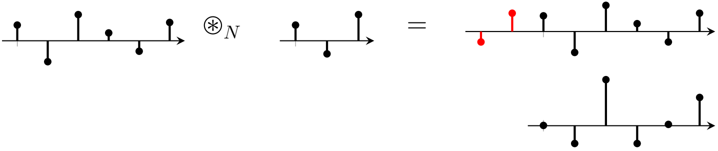
Convolution sum: $\sum_{k=0}^{M-1} h[k]x[k-n]$

thus $y[n]$ has only $M+N-1$ non-zero values.

EXPLANATION

Circular Convolution





Circular Convolution: $y \circledast_N [n] = \sum_{k=0}^{M-1} x[k]h[(n-k)_{\text{mod } N}]$

$y \circledast_N [n]$ has period $N \Rightarrow y \circledast_N [n]$ has N distinct values

Only the first $M-1$ values are affected by the circular repetition:
$y \circledast_N [n] = y[n]$ for $M-1 \leq n \leq N-1$

If we append $M-1$ zeros (or more) onto $x[n]$, then the circular repetition has no effect at all

You can also calculate the convolution is using a matrix, where the circular convolution $y[n] = x[n] * h[n]$ is the equivalent to:

A:

$$\begin{bmatrix} x[0] & x[3] & x[2] & x[1] \\ x[1] & x[0] & x[3] & x[2] \\ x[2] & x[1] & x[0] & x[3] \\ x[3] & x[2] & x[1] & x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix}$$

B:

$$\begin{bmatrix} x[0] & x[1] & x[2] & x[3] \\ x[1] & x[2] & x[3] & x[0] \\ x[2] & x[3] & x[0] & x[1] \\ x[3] & x[0] & x[1] & x[2] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix}$$

C:

$$\begin{bmatrix} x[0] & x[1] & x[2] & x[3] \\ x[1] & x[0] & x[3] & x[2] \\ x[2] & x[3] & x[0] & x[1] \\ x[3] & x[2] & x[1] & x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix}$$

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$$\begin{bmatrix} x[0] & x[1] & x[2] & x[3] \\ x[1] & x[2] & x[3] & x[0] \\ x[2] & x[3] & x[0] & x[1] \\ x[3] & x[0] & x[1] & x[2] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix}$$

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Circulant Matrix Method

One way to calculate the convolution is using a circulant matrix, where the circular convolution $y[n] = x[n] \circledast h[n]$ is the equivalent to:

$$\begin{bmatrix} x[0] & x[3] & x[2] & x[1] \\ x[1] & x[0] & x[3] & x[2] \\ x[2] & x[1] & x[0] & x[3] \\ x[3] & x[2] & x[1] & x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix}$$

Where $h[n]$ and $x[n]$ are made up of 4 samples.

Simple Trick Method

There is a also a very simple trick that can be used to calculate the convolution:

		-2	-1	0	1	2
			$x[-1]$	$x[0]$	$x[1]$	$x[2]$
		$h[-2]$	$h[-1]$	$h[0]$	$h[1]$	
			$h[1]x[-1]$	$h[1]x[0]$	$h[1]x[1]$	$h[1]x[2]$
		$h[0]x[-1]$	$h[0]x[0]$	$h[0]x[1]$	$h[0]x[2]$	$\times$
	$h[-1]x[-1]$	$h[-1]x[0]$	$h[-1]x[1]$	$h[-1]x[2]$	$\times$	
$h[-2]x[-1]$	$h[-2]x[0]$	$h[-2]x[1]$	$h[-2]x[2]$	$\times$		
$y[-3]$	$y[-2]$	$y[-1]$	$y[0]$	$y[1]$	$y[2]$	$y[3]$

Overlap save for $x[0:N-1]$ with $h[0:M-1]$ is defined as:

I: chop $x[n]$ into $\frac{N}{K}$ overlapping chunks of length $K+M-1$
II: $\circledast_{K+M-1}$ each chunk with $h[n]$
III: discard first $M-1$ from each chunk
IV: concatenate to make $y[n]$

What will be the output length of $y[n]$

A: $N+M-1$
B: $N-1$
C: N

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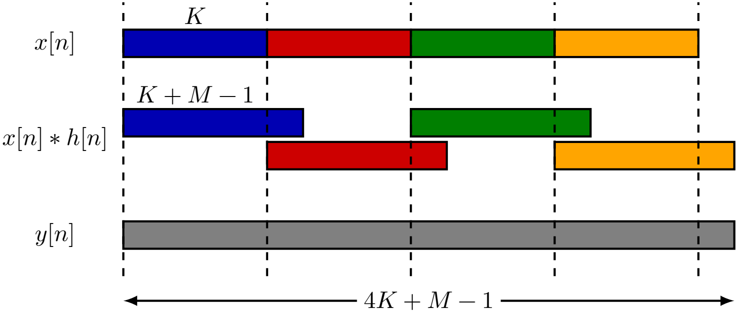
Overlap Add

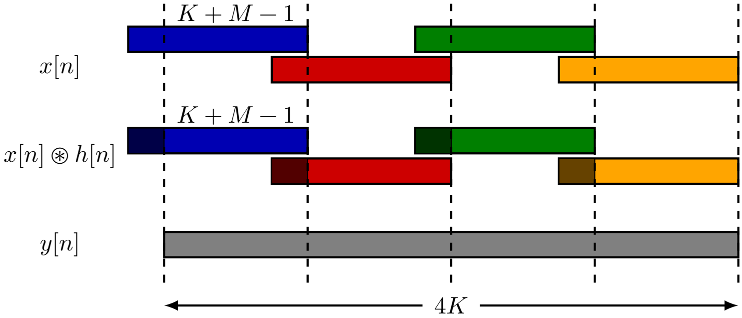


I: chop $x[n]$ into $\frac{N}{K}$ chunks of length K

II: convolve each chunk with $h[n]$

III: add up the results

Overlap Save



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If we convolve $x[n]$ with $h[n]$ then $x[n] * h[n] = \sum_{k=0}^{M-1} h[k]x[k-n]$.

What is the total arithmetic complexity ($\times$ or $+$ operations)?

A: $\approx MN$

B: $\approx 2MN$

C: $\approx 4MN$

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We know that we can perform convolution in the frequency-domain using a single DFT.

What is not a disadvantage of this approach?

A: More computation is required

B: No outputs until all $x[n]$ has been input

C: DFT may be very long if N is large

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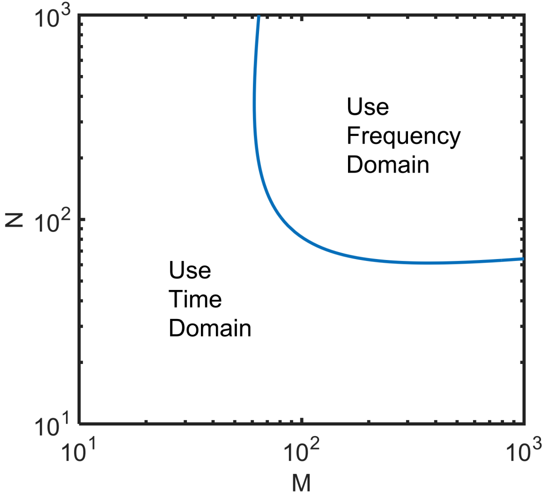

Example: $M = 10^3, N = 10^4$:

Direct: $2MN = 2 \times 10^7$ **with DFT:** = 1.8×10^6