Approximations (Series Again)

Why Taylor Series Matter

- Many functions (trigonometric, exponential, logarithmic) are difficult to work with directly.
- In engineering, we often only need an *approximation near a point*.
- Example: Real power in AC is: $P = VI\cos(\theta)$

  If the phase difference is small, then:
  - A useful approximation is $\cos(\theta) \approx 1 - \frac{\theta^2}{2}$
  - Makes equations dramatically simpler
- But why does this quadratic match so well near $\theta = 0$? This is exactly what Taylor series explain.

Idea: Replace Functions with Polynomials

- Polynomials are easy:
  - easy to differentiate
  - easy to integrate
  - easy to evaluate numerically
- Goal: approximate a function $f(x)$ by a polynomial that 'hugs' $f(x)$ near some point.
- Example: let's approximate $\cos(x)$ near 0 using a quadratic:

$$\cos(x) = c_0 + c_1 x + c_2 x^2$$

- Which c_0, c_1, c_2 make the polynomial resemble $\cos(x)$ as closely as possible near $x = 0$?

Step 1: Match the Value at the Point $x = 0$

- $\cos(0) = 1$ so we want $c_0 + c_1 x + c_2 x^2 = 1$
- Since $\cos(0) = c_0 + c_1(0) + c_2(0)^2 = c_0$, this forces:

$$c_0 = 1$$

- Therefore

$$\cos(x) = 1 + c_1 x + c_2 x^2$$

- The approximation and the function $\cos(x)$ now have the same value when $x = 0$.

Step 2: Match the Slope at the Point $x = 0$

- Slope of $\cos x$ at $x = 0$:

$$\cos'(x) = -\sin x, \quad \cos'(0) = 0$$

- Slope of the polynomial:

$$\cos'(x) = c_1 + 2c_2 x, \quad \cos'(0) = c_1 + 2c_2(0) = c_1$$

- For matching slopes:

$$c_1 = 0$$

- Therefore

$$\cos(x) = 1 + c_2 x^2$$

- The approximation and the function $\cos(x)$ now have the same value and slope when $x = 0$.

Step 3: Match the Curvature $\left(\frac{d^2y}{dx^2}\right)$

- Curvature of $\cos x$ at $x = 0$:

$$\cos''(x) = -\cos x, \quad \cos''(0) = -1.$$

- Curvature of the polynomial:

$$\cos''(x) = 2c_2, \quad \cos''(0) = 2c_2.$$

- Matching curvatures gives

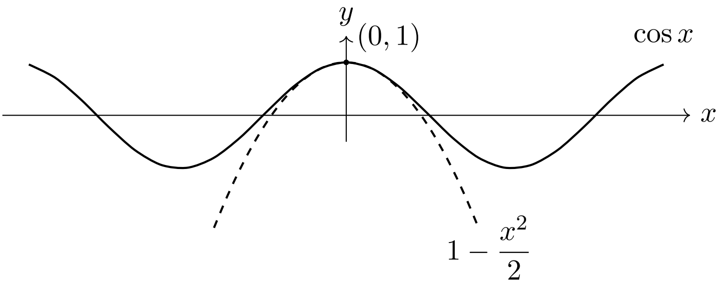
$$2c_2 = -1 \quad \Rightarrow \quad c_2 = -\frac{1}{2}.$$

- Hence the best quadratic approximation is

$$\cos(x) = 1 - \frac{x^2}{2}.$$

Visualising the Cosine Approximation

- Near $x = 0$, the graphs of $\cos x$ and $1 - \frac{x^2}{2}$ almost coincide.
- This is the idea of a Taylor polynomial: a polynomial that 'hugs' the graph of $f(x)$ near a point.



General Pattern for $\cos\theta$

- For $\cos\theta$, all the information near 0 is in its derivatives at 0:

$$\begin{aligned} \cos(0) &= 1, \\ \cos'(0) &= 0, \\ \cos''(0) &= -1, \\ \cos^{(3)}(0) &= 0, \\ \cos^{(4)}(0) &= 1, \\ &\vdots \end{aligned} \tag{1}$$

- All **odd** derivatives at 0 are 0; the **even** ones alternate between $+1$ and -1.

General Pattern for $\cos\theta$

- Notice that

$$\cos^{(n)}(0) = n! \times c_n \implies c_n = \frac{\cos^{(n)}(0)}{n!}$$

- So the Taylor series at 0 only has even powers:

$$\cos(x) = c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \cdots$$

where

$$\cos\theta = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

- In compact sigma notation:

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

General Pattern: Matching Higher Derivatives

- Note that each coefficient of a polynomial controls exactly one derivative at $x = 0$.
- Taking n derivatives of x^n produces $n!$

$$\frac{d^n}{dx^n}(x^n) = n!$$

- Thus the coefficient of x^n must be

$$\frac{f^{(n)}(0)}{n!}$$

- This yields the **Maclaurin series**.

The Maclaurin Series

- If $f(x)$ is sufficiently differentiable at 0, then

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \cdots$$

- More compact notation:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

- Each term ensures matching of successively higher derivatives at $x = 0$.

Example: Maclaurin Series of $\sin x$

- Derivatives repeat with cycle length 4:

$$\sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x, \ldots$$

- Evaluating at $x = 0$ yields:

$$0,\ 1,\ 0,\ -1,\ 0,\ 1, \ldots$$

- Therefore: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- In compact sigma notation:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

- **This approximation is excellent for small x!**

Example: Maclaurin Series of e^x

- Every derivative of e^x is e^x.
- Therefore

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \cdots$$

gives

$$e^x = e^x + xe^x + x^2\frac{e^x}{2!} + x^3\frac{e^x}{3!} + \cdots$$

- At $x = 0$, each derivative equals 1.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Standard series

- The Maclaurin series for commonly encountered expressions are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{5!} + \frac{x^8}{8!} + \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2^5}{15} + \dots \qquad -\pi/2 < x < \pi/2$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots \qquad -1 < x \le 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Standard series

- Hyperbolic trigonometric expressions

$$\sinh x = \quad x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\cosh x = \quad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

Standard Series. The binomial series

- The same method can be applied to obtain the binomial expansion

$$(1+x)^n = 1 + nx + \frac{x^2}{2!}n(n-1) + \frac{x^3}{3!}n(n-1)(n-2) + \dots$$

for $-1 < x < 1$

Approximate values

- The Maclaurin series expansions can be used to find approximate numerical values of expressions. For example, evaluate $\sqrt{1.02}$ to 5 decimal places.
- Use the binomial expansion

$$(1+x)^n = 1 + nx + \frac{x^2}{2!}n(n-1) + \frac{x^3}{3!}n(n-1)(n-2) + \dots$$

- in this case $x = 0.02$ and $n = \frac{1}{2}$ then

$$\begin{aligned}
(1+0.02)^{\frac{1}{2}} &= 1 + \frac{1}{2} \times (0.02) + \frac{0.02^2}{2!}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) \\
&\quad + \frac{0.02^3}{3!}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right) + \dots \\
&= 1 + 0.01 - 0.0005 + 0.0000005 + \dots \\
&= 1.0099505
\end{aligned}$$

Convergence of Series

- Some Maclaurin series converge for all x (e.g., $e^x, \sin x, \cos x$).
- Others converge only for a limited range, for example:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

  only converges only for $|x| < 1$.

- Derivative information at one point only works while the function remains well-behaved.
- If the function misbehaves the series stops working before you reach that point.

Approximating Around Point $x = a$

- Maclaurin Series is actually just a Taylor Series about $a = 0$.
- Maclaurin's series:

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} \dots$$

gives the expansion of $f(x)$ about the point $x = 0$.

- To expand about the point $x = a$, **Taylor's series** is employed:

$$f(x + a) = f(a) + xf'(a) + x^2\frac{f''(a)}{2!} + x^3\frac{f'''(a)}{3!} \dots$$

Taylor's series

- To expand about the point $x = a$, Taylor's series can be derived from the Maclaurin's series using the function
$$F(x) = f(x + a)$$
- Then $F(0) = f(a)$, $F'(0) = f'(a)$ etc. and
$$F(x) = F(0) + xF'(0) + x^2\frac{F''(0)}{2!} + x^3\frac{F'''(0)}{3!}\ldots$$
gives
$$f(x + a) = f(a) + xf'(a) + x^2\frac{f''(a)}{2!} + x^3\frac{f'''(a)}{3!}\ldots$$

General Taylor series

- General Taylor series:

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2\frac{f''(a)}{2!} + (x-a)^3\frac{f'''(a)}{3!} + \cdots$$

- Derived by applying Maclaurin to $F(x) = f(x+a)$.

Limiting values – indeterminate forms

- Sometimes, we have to find the limiting value of a function of x when $x \to 0$, or perhaps when $x \to a$.

$$\lim_{x\to 0}\left(\frac{x^5+5x-14}{x^2-5x+8}\right)=\frac{0+0-14}{0-0+8}=-\frac{14}{8}$$

- However, it is not always that straightforward.

$$\lim_{x\to 2}\left(\frac{x^2+5x-14}{x^2-5x+6}\right)=\frac{4+10-14}{4-10+6}=\frac{0}{0}$$

- $0/0$!

Limiting values – indeterminate forms

- Power series expansions can sometimes be employed to evaluate the limits of indeterminate forms.
- For example

$$\lim_{x\to 0}\left(\frac{\tan x - x}{x^3}\right) = \lim_{x\to 0}\left(\frac{x + x^3/3 + O(x^5) - x}{x^3}\right)$$

- where we have expanded the $\tan$ function to third order (x^3), the notation $O(x^5)$ means that all the next terms in the expansion are of order 5 or larger (that is x^5, x^6, ..., etc.)
- taking the limit

$$\lim_{x\to 0}\left(\frac{x + x^3/3 + O(x^5) - x}{x^3}\right) = \lim_{x\to 0}\left(\frac{1}{3} + O(x^2)\right) = \frac{1}{3}$$

Limiting values – indeterminate forms

- The general procedure is:
  - Express the given function in terms of power series
  - Simplify the function as far as possible.
  - Then determine the limiting value – which should now be possible.
- However, there may well be occasions when direct substitution gives the indeterminate form of and we do not know the series expansion of the function concerned. What are we going to do then?

Limits and Indeterminate Forms

- A limit describes what a quantity *approaches* as the input gets arbitrarily close to some value.
- Many expressions behave nicely as $x \to a$, even if the function is not defined at a.
- Example:

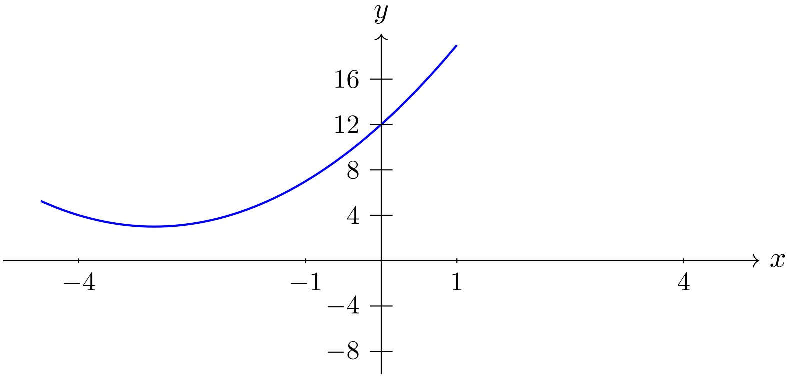
$$y = \frac{(2+x)^3 - 8}{x}.$$

- Although plugging in $x = 0$ gives $0/0$, the surrounding values clearly approach 12.

$$\begin{aligned}\lim_{x\to 0} \frac{(2+x)^3 - 8}{x} &= \lim_{x\to 0} \frac{x^3 - 6x^2 + 12x}{x} \\ &= \lim_{x\to 0}(x^2 - 6x + 12) = 12\end{aligned}$$

Geometric Meaning

- Lets plot $y = \dfrac{(2+x)^3 - 8}{x}$



L'Hôpital's Rule - Recall Differentiation

- One of the most important limits is that of differentiation

$$\frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- The limit is need because when $h = 0$, both numerator and denominator equal 0.

L'Hôpital's Rule - Concept

- Let's say we have a function $y = \frac{f(x)}{g(x)}$ where $g(x)$ and $f(x)$ both cross the x-axis at $x = a$
- Therefore $\frac{f(a)}{g(a)} = \frac{0}{0}$
- so what is

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \ ?$$

- A tiny nudge dx to the input gives:

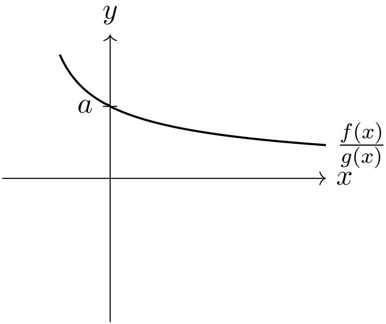
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\frac{df}{dx}(a)dx}{\frac{df}{dx}(a)dx} = \frac{\frac{df}{dx}(a)}{\frac{df}{dx}(a)}$$

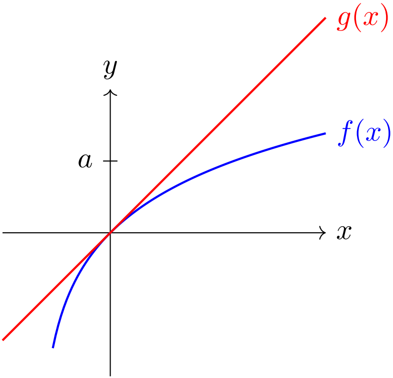
- Recall from previous slide:

$$\frac{df}{dx}(a)dx = \lim_{dx \to 0} f(a + dx) - f(a).$$

L'Hôpital's Rule - Visualized

$$\frac{f(x)}{g(x)} = \frac{\ln(1+x)}{x}$$





L'Hôpital's Rule (Formal Statement)

- Suppose $f(x)$ and $g(x)$ satisfy:

$$f(a) = g(a) = 0 \quad \text{or} \quad |f(x)|, |g(x)| \to \infty \text{ as } x \to a.$$

- If $f'(x)$ and $g'(x)$ exist near a and $g'(x) \neq 0$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

- L'Hôpital doesn't *solve* the limit - it transforms it into a simpler one.

Example

- Consider:

$$\frac{\sin(\pi x)}{x^2 - 1}.$$

- At $x = 1$:

$$\sin(\pi) = 0, \qquad 1^2 - 1 = 0 \quad \Rightarrow \quad 0/0.$$

- Apply L'Hôpital:

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x^2 - 1} = \lim_{x \to 1} \frac{\pi \cos(\pi x)}{2x}.$$

- Evaluate at $x = 1$:

$$\frac{\pi \cos(\pi)}{2} = \frac{\pi(-1)}{2} = -\frac{\pi}{2}.$$

Example

- Compute:

$$\lim_{x \to 1} \frac{x^3 + x^2 - x - 1}{x^2 + 2x - 3}.$$

- See that:

$$f(1) = g(1) = 0.$$

- Apply L'Hôpital:

$$\lim_{x \to 1} \frac{3x^2 + 2x - 1}{2x + 2} = \frac{4}{4} = 1.$$