Series

Recap of sequences

- a collection of objects where order matters and repetitions are significant
- Alternatively, we may think of a sequence as a function with integer domain.
- A finite sequence contains only a finite number of terms.
- An infinite sequence is unending.

Series

- A series may be considered in terms of the partial sums of the corresponding sequence.

- If $u_0, u_1, u_2, \ldots$ is a sequence, then the partial sums are

$$
\begin{aligned}
S_0 &= u_0 \\
S_1 &= u_0 + u_1 \\
S_2 &= u_0 + u_1 + u_2 \\
&\ldots \\
S_n &= u_0 + u_1 + u_2 + \ldots + u_n = \sum_{r=0}^{n} u_r
\end{aligned}
$$

- Then what is formed is called a series – a sum of terms of a sequence.

Series

- The sequence $u_0, u_1, u_2, \ldots$ is **summable** if the sequence of partial sums $S_0, S_1, S_2, \ldots$ converges, and

$$\sum_{r=0}^{\infty} u_r = \lim_{n\to\infty} \left(\sum_{r=0}^{n} u_r \right) = \lim_{n\to\infty} S_n$$

- As there are many types of sequence, so there are many types of series.

Arithmetic Series

- An arithmetic sequence is defined as

$$u_n = a + nd$$

  (starting at $n = 0$).
- a is the first term and d is the common difference.
- The arithmetic series S_n (sum of u_0 to u_n) is

$$S_n = \frac{n+1}{2}(2a + nd)$$

- As an example, consider the sum

$$S_{99} = 1 + 2 + 3 + \ldots + 100 \quad \text{(100 terms)}$$

- where $a = 1$ and $d = 1$.

$S = 1 + 2 + 3 + \ldots + 100$

- The sum is $S = 1 + 2 + 3 + \ldots + 100$ that we can rearrange as
- $S = 100 + 99 + 98 + \ldots + 1$
  that is from the largest value to the smallest value
- Now we add together these two sums

$$\begin{array}{cccccccc} S & = & 1 & +2 & +3 & \ldots & +100 \\ + & & & & & & \\ S & = & 100 & +99 & +98 & \ldots & +1 \\ \hline 2S & = & 101 & +101 & +101 & \ldots & +101 \end{array}$$

- which gives
  $2S = 100 \times 101$ that is

$$S_{99} = \frac{10100}{2} = 5050$$

- $S_n = \frac{(n+1)\times(n+2)}{2}$, therefore, $S_{99} = \frac{100\times 101}{2}$

Arithmetic Series

- Using the trick when summing $S = 1 + 2 + \ldots + 100$ but now for the general arithmetic series
- Given
$$u_n = a + nd.$$
- Show that:
$$S_n = \frac{n+1}{2}(2a + nd).$$

Arithmetic Series

- Arrange forwards and backwards:

$$
\begin{array}{ccccccc}
S_n & = & a & +(a+d) & \dots & +(a+nd) \\
+ & & & & & \\
S_n & = & (a+nd) & +(a+(n-1)d) & \dots & +a \\
\hline
2S_n & = & (2a+nd) & +(2a+nd) & \dots & +(2a+nd)
\end{array}
$$

- There are $n+1$ equal terms:

$$2S_n = (n+1)(2a+nd)$$

$$S_n = \frac{n+1}{2}(2a+nd)$$

Some results

- The **arithmetic mean** of P and Q is the number A such that

$$P + A + Q$$

- are terms of an arithmetic series.
- So we have that $A - P = d$ and $Q - A = d$ hence $A - P = Q - A$ that gives $2A = P + Q$ then

$$A = \frac{P + Q}{2}$$

- the arithmetic mean of two numbers is their *average*

Some results

- The three arithmetic means between two numbers P and Q are the numbers A, B, and C then

$$P + A + B + C + Q$$

- is an arithmetic series. Then

$$d = \frac{Q - P}{4}$$

- then $A = P + (Q - P)/4$, $B = P + (Q - P)/2$ and $C = P + 3(Q - P)/4$

Geometric series

- A geometric sequence is

$$u_n = ar^n.$$

(starting at $n = 0$).

- where a is the first term and r is the common ratio.
- The geometric series is defined as

$$S_n = \sum_{r=0}^{n} ar^r = \frac{a(1 - r^{n+1})}{1 - r}.$$

Geometric Series

- To get the general form of the geometric series, we are going to use a similar trick as in the arithmetic series.

- Subtract rS_n from S_n:

$$\begin{array}{cccccccc}
S_n & = & a & +ar & +ar^2 & \dots & +ar^n & \\
- & & & & & & & \\
rS_n & = & & +ar & +ar^2 & \dots & +ar^n & +ar^{n+1} \\
\hline
S_n - rS_n & = & a & +0 & +0 & \dots & +0 & -ar^{n+1}
\end{array}$$

- Then $S_n - rS_n = a - ar^{n+1}$ or $S_n(1-r) = a(1-r^{n+1})$

$$S_n = \frac{a(1-r^{n+1})}{1-r}$$

Some results

- The geometric mean of P and Q is the number A such that
$$P + A + Q$$
- are terms of an geometric series.
- So we have that $A/P = r$ and $Q/A = r$ hence $A/P = Q/A$ that gives $A^2 = PQ$ then
$$A = \sqrt{PQ}$$
- the geometric mean of two numbers is the *square root of their product*

Some results

- The three geometric means between two numbers P and Q are the numbers A, B, and C then

$$P + A + B + C + Q$$

- is a geometric series. Then $Q/P = r^4$ or $r = (Q/P)^{1/4}$
- then $A = P(Q/P)^{1/4}$, $B = P(Q/P)^{2/4}$ and $C = P(Q/P)^{3/4}$

Series of powers of the natural numbers

- Consider

$$0 + 1 + 2 + 3 + \ldots + n = \sum_{r=0}^{n} r.$$

- This is arithmetic ($a = 0$, $d = 1$):

$$\sum_{r=0}^{n} r = \frac{n+1}{2}(2a + nd) = \frac{(n+1)n}{2}$$

Sum of squares

- Compute:

$$0^2 + 1^2 + 2^2 + \ldots + n^2 = \sum_{r=0}^{n} r^2$$

- Result:

$$\sum_{r=0}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Derivation of $\sum_i^n r^2$

- First notice $(r+1)^3 = r^3 + 3r^2 + 3r + 1$ so r^2 can be written as

$$r^2 = \frac{(r+1)^3 - r^3 - 3r - 1}{3}$$

- write the sum using this

$$\begin{aligned}
\sum_{r=0}^{n} r^2 &= \frac{1}{3}\sum_{r=0}^{n}\left((r+1)^3 - r^3 - 3r - 1\right) \\
&= \frac{1}{3}\left(\sum_{r=0}^{n}(r+1)^3 - \sum_{r=0}^{n} r^3 - \sum_{r=0}^{n} 3r - \sum_{r=0}^{n} 1\right)
\end{aligned}$$

Derivation of $\sum_i^n r^2$

- Doing the sums
  - $\sum_{r=0}^n 1 = n+1$ as $\sum_{r=1}^n 1$ is summing 1, $n+1$ times
  - $\sum_{r=0}^n r = ((n+1)n/2)$ from previous result
- now these two sums together $\sum_{r=1}^n (r+1)^3 - \sum_{r=1}^n r^3$

$$
\begin{array}{rcccccccc}
\sum_{r=0}^n (r+1)^3 & = & & 1 & +2^3 & \dots & +n^3 & +(n+1)^3 \\
- & & & & & & & \\
\sum_{r=0}^n r^3 & = & 0 & +1 & +2^3 & \dots & +n^3 & \\
\hline
 & = & 0 & +0 & +0 & \dots & +0 & +(n+1)^3
\end{array}
$$

- then

$$\sum_{r=0}^n (r+1)^3 - \sum_{r=0}^n r^3 = (n+1)^3$$

Derivation of $\sum_i^n r^2$

- Putting the results together

$$
\begin{aligned}
\sum_{r=0}^{n} r^2 &= \frac{1}{3}\left(\sum_{r=0}^{n}(r+1)^3 - \sum_{r=0}^{n} r^3 - \sum_{r=0}^{n} 3r - \sum_{r=0}^{n} 1\right) \\
&= \frac{1}{3}\left((n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)\right) \\
&= \frac{1}{3}\left(n^3 + 3n^2 + 3n + 1 - \frac{3n(n+1)}{2} - n - 1\right) \\
&= \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right) = \frac{n}{6}(2n^2 + 3n + 1) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

Sum of cubes

- In this case

$$\sum_{r=0}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$$

- can you solve it? Hint use that

$$(r+1)^4 = r^4 + 4r^3 + 6r^2 + 4r + 1$$

Infinite series

- An infinite series is one whose terms continue indefinitely.

  For example, the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

- is a geometric sequence where $a = 1$ and $r = \frac{1}{2}$, giving rise to the partial sum

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}.$$

- Then

$$S_n = \frac{a(1 - r^{n+1})}{1 - r} = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

Infinite series

- As n increases without bound, $1/2^n$ decreases and approaches 0.
- The partial sums are

$$S_n = 2\left(1 - \frac{1}{2^{n+1}}\right).$$

- As $n \to \infty$, $\frac{1}{2^{n+1}} \to 0$, so as $n \to \infty$

$$S_n \to 2.$$

- We say that the limit of S_n as n approaches infinity is

$$\lim_{n\to\infty} S_n = S_\infty = 2.$$

Infinite series

- Note that when it is stated that the limit of S_n as n approaches infinity is 2,

$$\lim_{n\to\infty} S_n = S_\infty = 2$$

- what is meant is that a value of S_n can be found as close to the number 2 as we wish by selecting a sufficiently large enough value of n.

- However, S_n never actually attains the value of 2 in this case.

Example

- For the geometric series
$$S_n = \frac{a(1-r^{n+1})}{1-r},$$
- when $|r| < 1$, we have $r^{n+1} \to 0$ as $n \to \infty$.
- Therefore
$$\lim_{n\to\infty} S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$
- Hence, for the example above where $a = 1$ and $r = \frac{1}{2}$,
$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$
$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

No limit

- Sometimes a series has no limit. For example, the sequence
$$1, 3, 5, \ldots$$
- is an arithmetic sequence where $u_n = 2n + 1$ for $n \geq 0$, with $a = 1$ and $d = 2$. The partial sums are
$$\begin{aligned} S_n &= \sum_{r=0}^{n} (2r + 1) = 1 + 3 + 5 + \cdots + (2n + 1) \\ &= \frac{n+1}{2}(2 + 2n) = (n + 1)^2 \end{aligned}$$
- As $n \to \infty$ so $(n + 1)^2 \to \infty$ and $S_n = (n + 1)^2 \to \infty$.
- That is
$$\lim_{n \to \infty} S_n = S_\infty = \infty.$$

Indeterminate form

- The fact that as $n \to \infty$ so $1/n \to 0$ can be usefully employed to find the limits of certain indeterminate forms.
- For example

$$\lim_{n\to\infty} \frac{5n+3}{2n-7} = \lim_{n\to\infty} \frac{5+\frac{3}{n}}{2-\frac{7}{n}} = \frac{5+0}{2-0} = \frac{5}{2}$$

Convergent and divergent series

- An infinite series whose partial sums tend to a finite limit is said to be a convergent series.
  If an infinite series does not converge then it is said to diverge.
- If a formula for S_n cannot be found it may not be possible by simple inspection to decide whether or not a given series converges.
- To help with this, we introduce some convergence tests.

Test of convergence

Test 1: A series cannot converge unless its terms ultimately tend to zero

Test 2: The comparison test

Test 3: D'Alembert's ratio test for positive terms

Test 1

- If
$$S_n = u_0 + u_1 + u_2 + \ldots + u_n,$$
- then the series can only converge if
$$\lim_{n\to\infty} u_n = 0.$$
- Notice that this *does not mean* that if $\lim_{n\to\infty} u_n = 0$ then S_n converges.
- Example: the harmonic series
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$
diverges, even though $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0.$

Example continuation

- To show this, we can group the terms as follows

$$\sum_{r=0}^{\infty} \frac{1}{r+1} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

- Now we look for a bound, $\left(\frac{1}{3} + \frac{1}{4}\right) > \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}$ and $\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = \frac{1}{2}$
- then

$$\sum_{r=0}^{\infty} \frac{1}{r+1} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \dots = \infty$$

- then

$$\sum_{r=0}^{\infty} \frac{1}{r+1} = \infty$$

Test 2: The comparison test

- Let $\sum a_n$ and $\sum b_n$ be series with positive terms.
- **Convergence:** If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ also converges.
- **Divergence:** If $0 \leq b_n \leq a_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ also diverges.
- A standard comparison family is the p-series:

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots \frac{1}{n^p} + \ldots = \sum_{r=0}^{\infty} \frac{1}{(r+1)^p},$$

which converges if $p > 1$ and diverges if $p \leq 1$.

Test 2: Example

- Consider the series $\sum_{n=1}^{\infty} \dfrac{1}{n^3+n}$
- For all $n \geq 1, \quad n^3 + n \geq n^3 \quad \Rightarrow \quad \dfrac{1}{n^3+n} \leq \dfrac{1}{n^3}$
- The comparison series is
$$\sum_{n=1}^{\infty} \frac{1}{n^3},$$
a p-series with $p = 3 > 1$, which is convergent.
- Therefore, by the comparison test,
$$\sum_{n=1}^{\infty} \frac{1}{n^3+n} \quad \text{converges}$$

Test 3: D'Alembert's ratio test for positive series

- If

$$u_0 + u_1 + u_2 + \ldots + u_n + \ldots = \sum_{r=0}^{\infty} u_r$$

  is a series of positive terms, then

  - if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, the series converges,
  - if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$, the series diverges,
  - if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, the test is inconclusive.

Question

- Determine whether the following series is convergent or divergent:
$$1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \dots$$
- We can write the nth term as
$$u_n = \frac{2n+1}{2^n}, \quad n = 0, 1, 2, \dots$$
- Then the next term is
$$u_{n+1} = \frac{2(n+1)+1}{2^{n+1}} = \frac{2n+3}{2^{n+1}}.$$
- The ratio is
$$\frac{u_{n+1}}{u_n} = \left(\frac{2n+3}{2^{n+1}}\right)\left(\frac{2^n}{2n+1}\right) = \frac{1}{2}\left(\frac{2n+3}{2n+1}\right).$$

Question. Continue

- The limit of the ratio is

$$\begin{aligned}\lim_{n\to\infty}\frac{u_{n+1}}{u_n} &= \lim_{n\to\infty}\left(\frac{1}{2}\cdot\frac{2n+3}{2n+1}\right)\\ &= \frac{1}{2}\lim_{n\to\infty}\frac{2+\frac{3}{n}}{2+\frac{1}{n}}=\frac{1}{2}\cdot\frac{2}{2}=\frac{1}{2}.\end{aligned}$$

- Since this limit is < 1, the series converges by the ratio test.
- Note that the limit $\frac{1}{2}$ here is the limit of the *ratio*, not the sum of the series. (In fact, the sum of the series is 6.)

Absolute convergence

- If a series $\sum_{r=0}^{\infty} u_r$ converges, then
- the series of absolute values of the terms

$$\sum_{r=0}^{\infty} |u_r|$$

- may or may not converge.
- If a series converges and the series of absolute values of the terms also converges, then the series is said to be **absolutely convergent**.
- In fact, if $\sum_{r=0}^{\infty} |u_r|$ converges, then $\sum_{r=0}^{\infty} u_r$ also converges.

Convergent test again

- If

$$u_0 + u_1 + u_2 + \ldots + u_n + \ldots = \sum_{r=0}^{\infty} u_r$$

is a series where the terms can be positive or negative, then

  - if $\displaystyle\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} < 1$, the series converges,
  - if $\displaystyle\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} > 1$, the series diverges,
  - if $\displaystyle\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = 1$, the test is inconclusive.

Conditionally convergent

- If a series $\sum_{r=0}^{\infty} u_r$ converges, but
- the series $\sum_{r=0}^{\infty} |u_r|$ diverges, then

$$\sum_{r=0}^{\infty} u_r$$

- is said to be **conditionally convergent.**

Example: alternating harmonic series

- Consider the alternating harmonic series
$$\sum_{r=0}^{\infty} u_r \quad \text{with} \quad u_r = \frac{(-1)^r}{r+1}.$$
- Written out:
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$
- The terms satisfy Test 1: $\lim_{r\to\infty} u_r = 0$, so the series may converge.

- Using more advanced techniques beyond this course, it can be in fact shown that
$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} = \ln 2.$$

Conditional convergence

- Now consider the series of absolute values:

$$\sum_{r=0}^{\infty} |u_r| = \sum_{r=0}^{\infty} \frac{1}{r+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

- The harmonic series, which we have shown diverges.
- Therefore:

$$\sum_{r=0}^{\infty} |u_r| = \infty \qquad \text{(diverges)},$$

  but

$$\sum_{r=0}^{\infty} u_r = \ln 2 \qquad \text{(converges)}.$$

- So the alternating harmonic series is **conditionally convergent**.