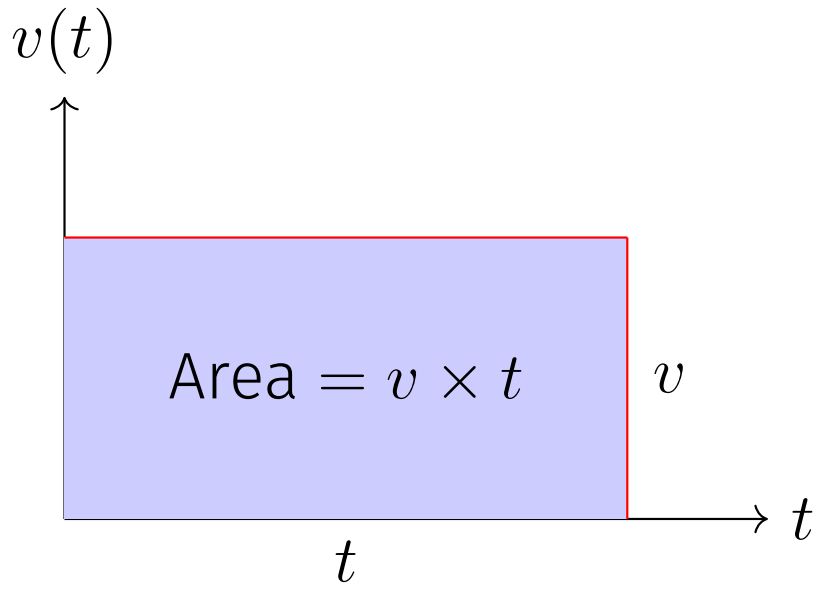
Integration

Integration: The Intuition

- Integration can be understood as the inverse of differentiation.
- Example context: You can see only the plane's speedometer; how far have you travelled?
- If you know velocity $v(t)$, distance travelled is the area under the velocity-time graph.
- Goal: Build intuition that *integrals accumulate small changes*.

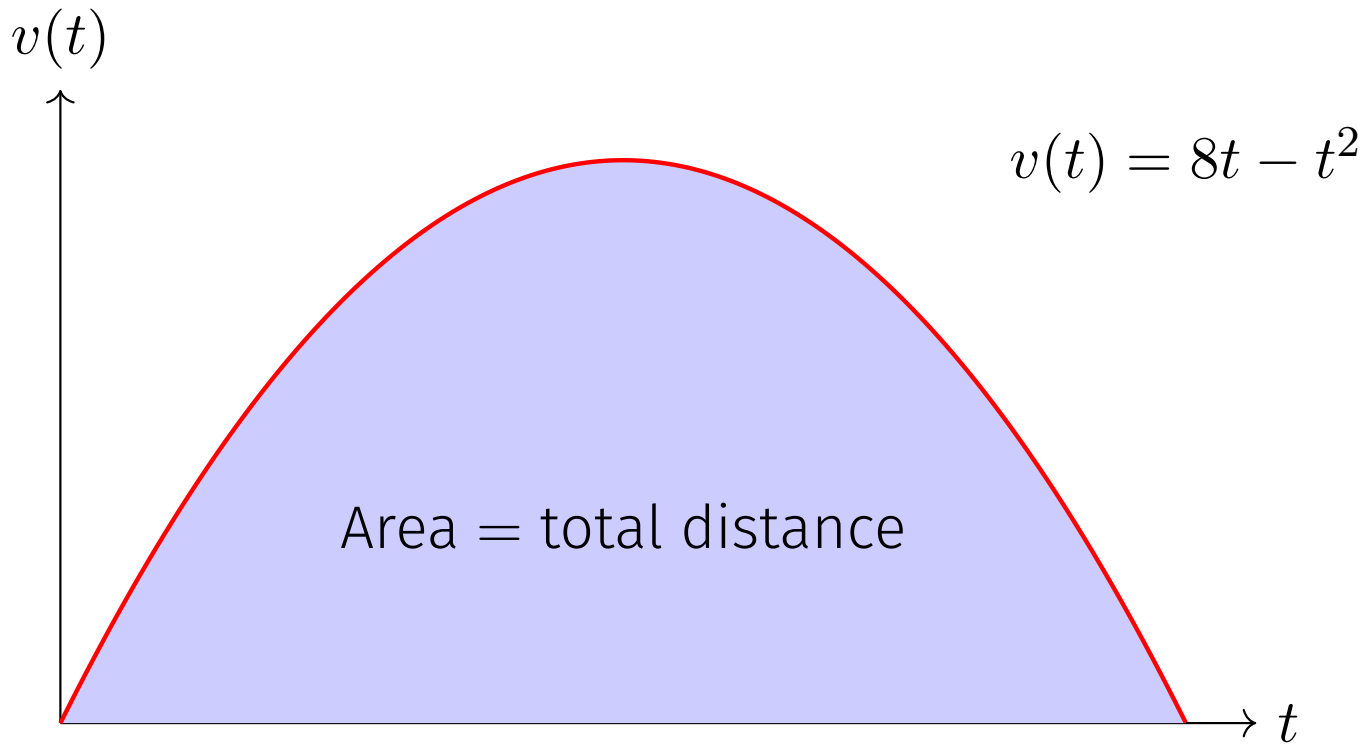
Distance from Velocity

- For constant velocity: distance $= v \times t$



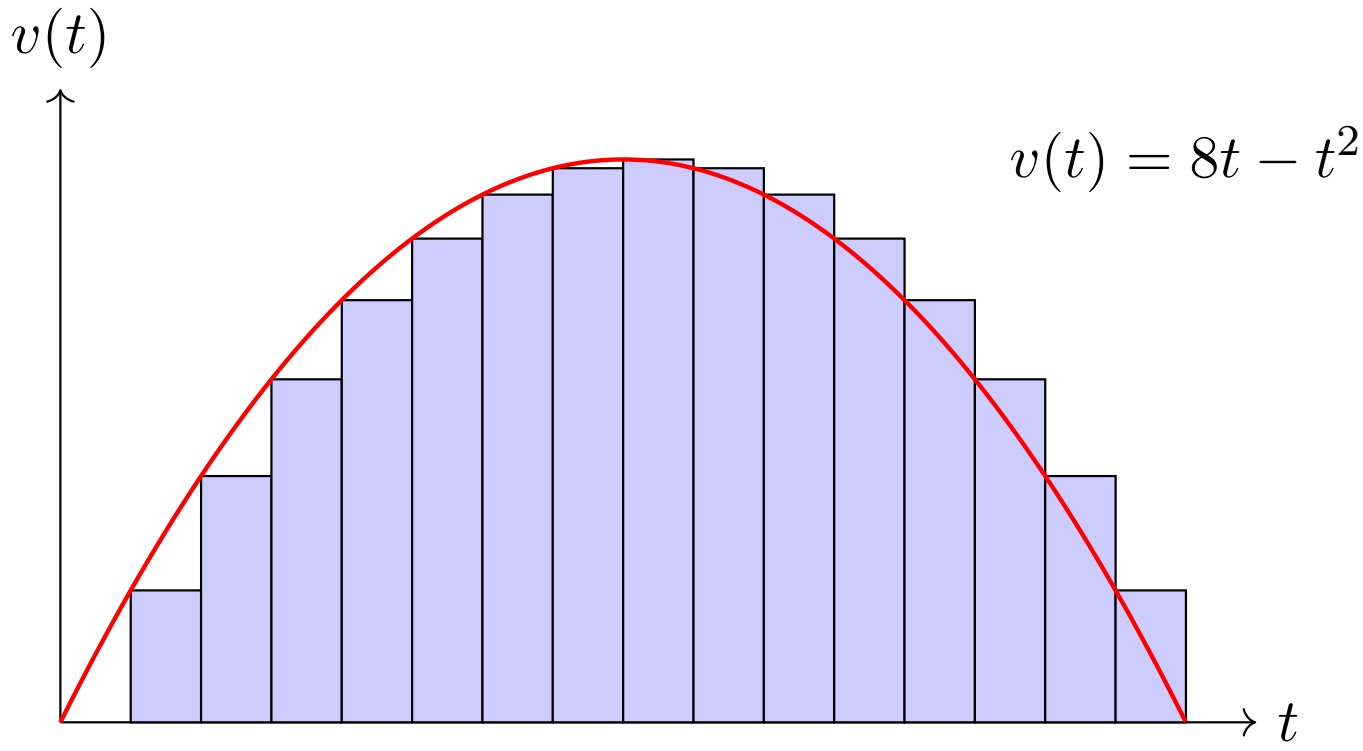
- When velocity varies, distance is the total area under a curve.

Area Under the Curve $v(t) = 8t - t^2$

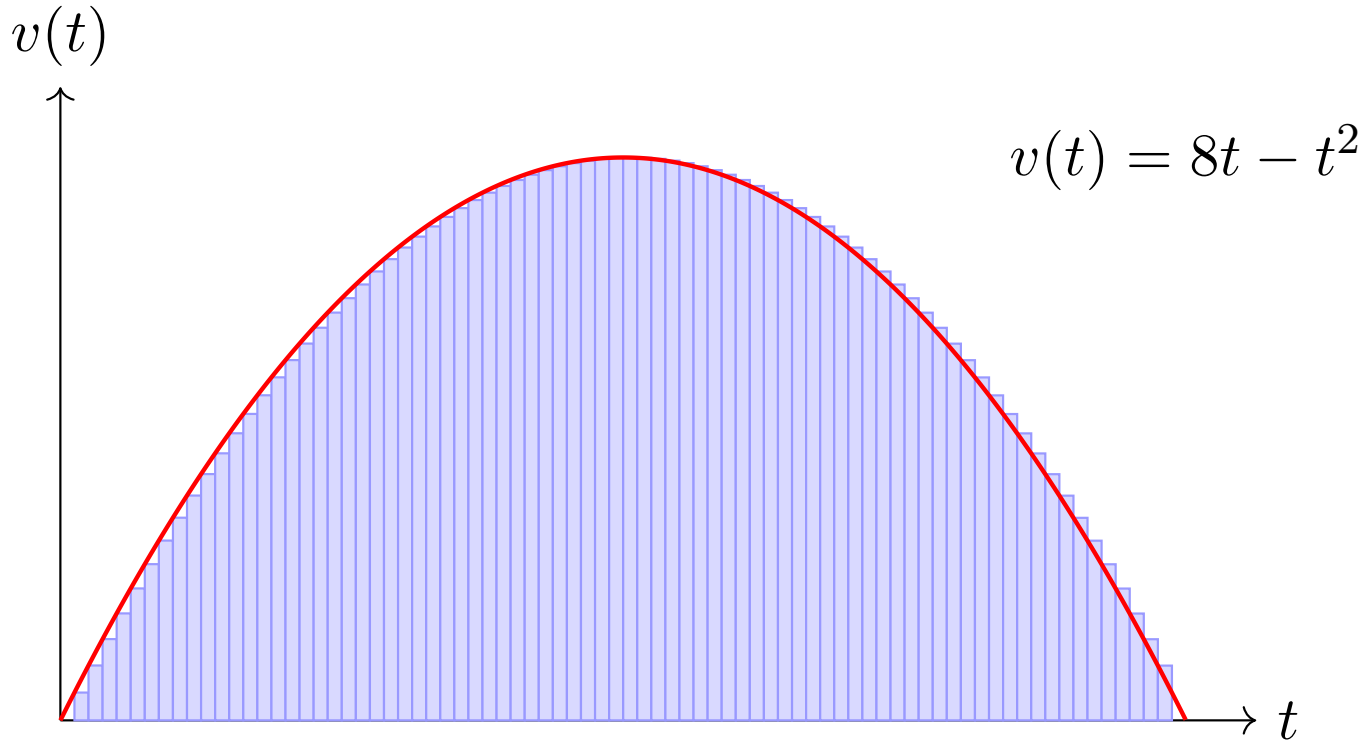


Approximating the Area

- Divide time into small pieces of width Δt.
- Approximate velocity as constant on each tiny interval.



Approximating the Area with Many Rectangles



- A smaller Δt gives a much finer approximation
- More rectangles $\rightarrow$ better estimate

Approximating the Area

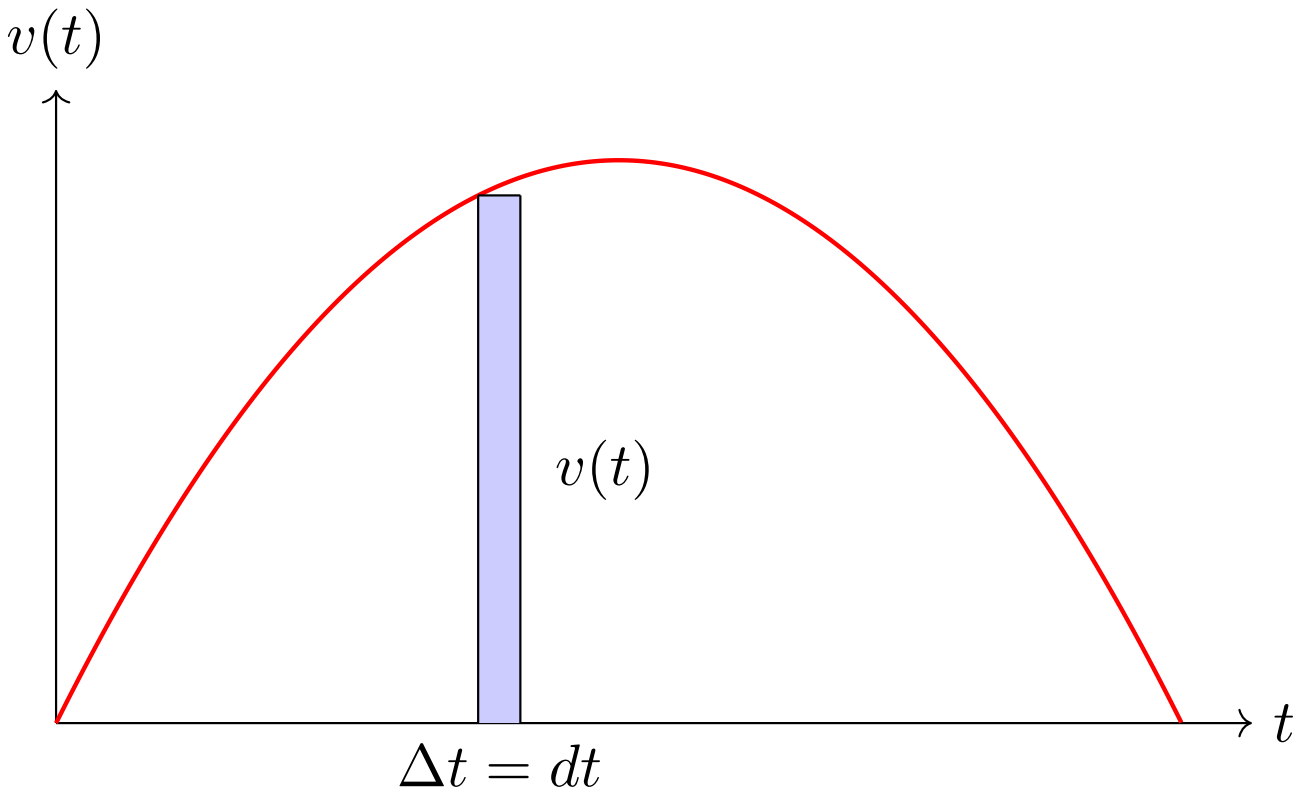
- Each rectangle has height $v(t)$ and width Δt
- Distance on each piece:

$$\Delta A \approx v(t)\,\Delta t$$

- Total distance is approximately the sum of N rectangles:

$$A \approx \sum_{n=0}^{N-1} v(t)\,\Delta t$$

Derivative of the Area-So-Far Function



- Therefore

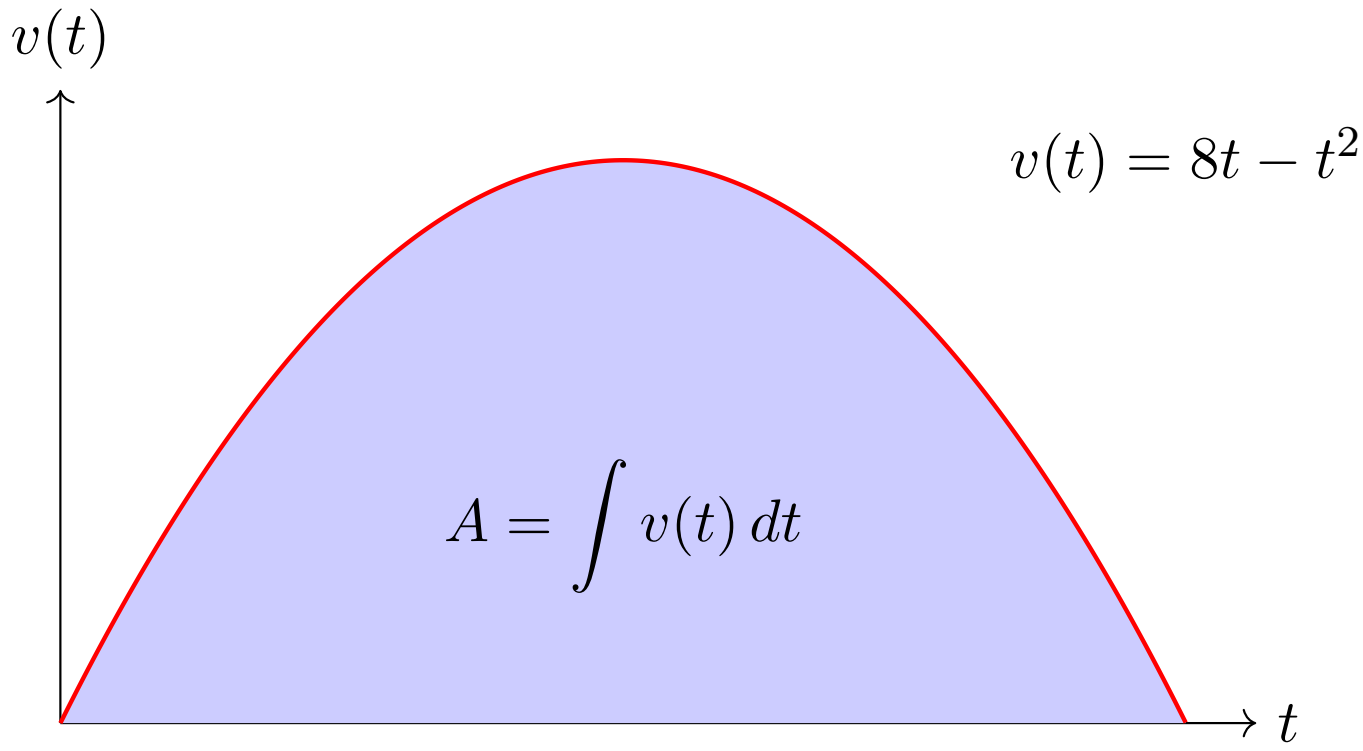
$$A \approx \sum_{n=0}^{N-1} v(t)\,dt \quad \Longrightarrow \quad A = \underbrace{\int v(t)\,dt}_{dt \to 0}$$

From Sum to Integral

- As $dt \to 0$, the approximation becomes exact
- The limit is called the *integral*:

$$\int_0^T v(t)\,dt = \text{area under the curve}$$

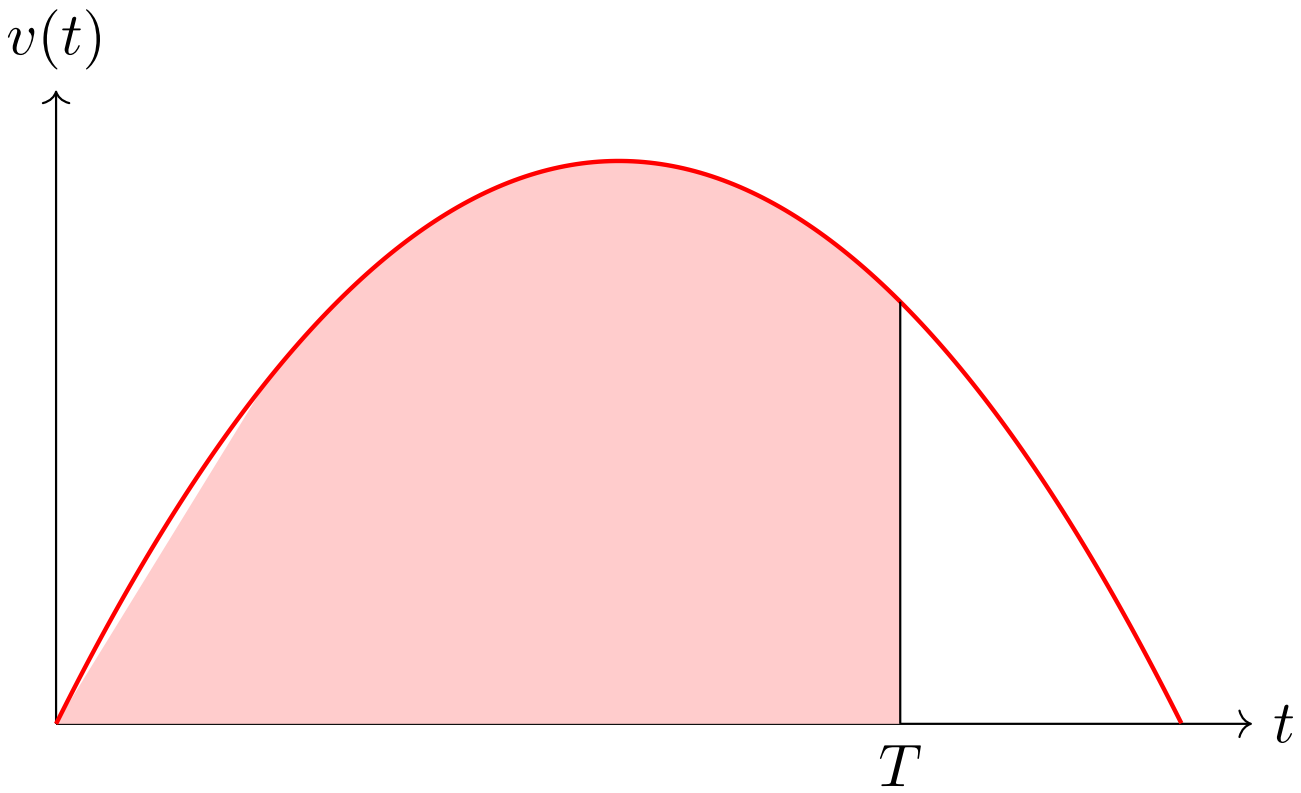
Area Under the Curve $v(t) = 8t - t^2$



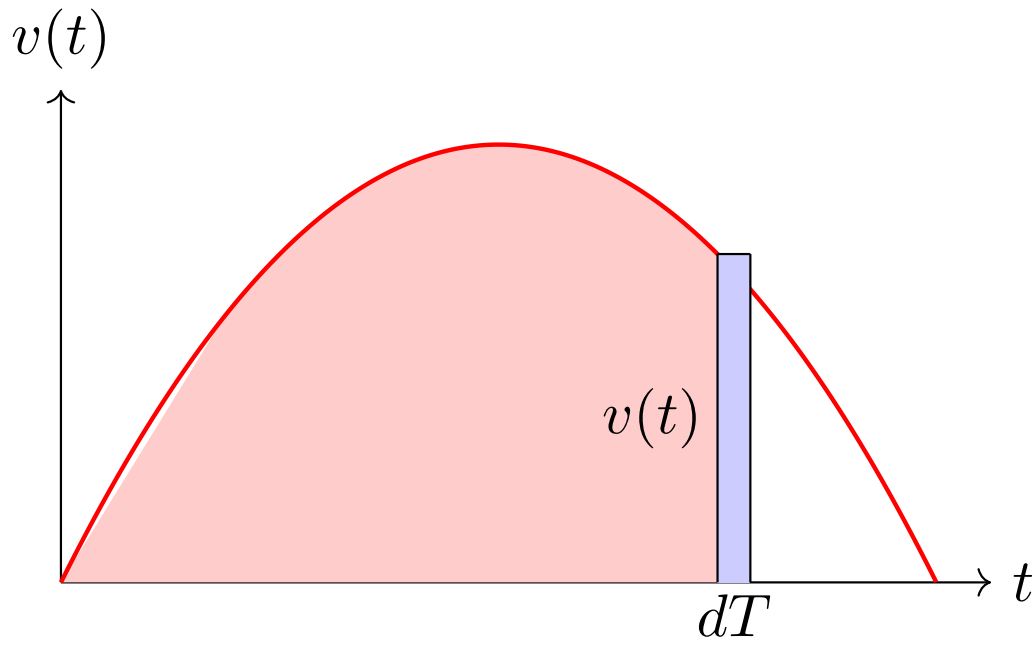
An Antiderivative Appears

- Treat the upper limit T as a variable giving the area function:

$$A(T) = \int_0^T v(t)\,dt$$



An Antiderivative Appears



- It can be seen that, $dA = dT \times v(t)$
- Thus, $A(T)$ is an *antiderivative* of $v(t)$

$$\frac{dA}{dT} = v(t)$$

Antiderivative of Example

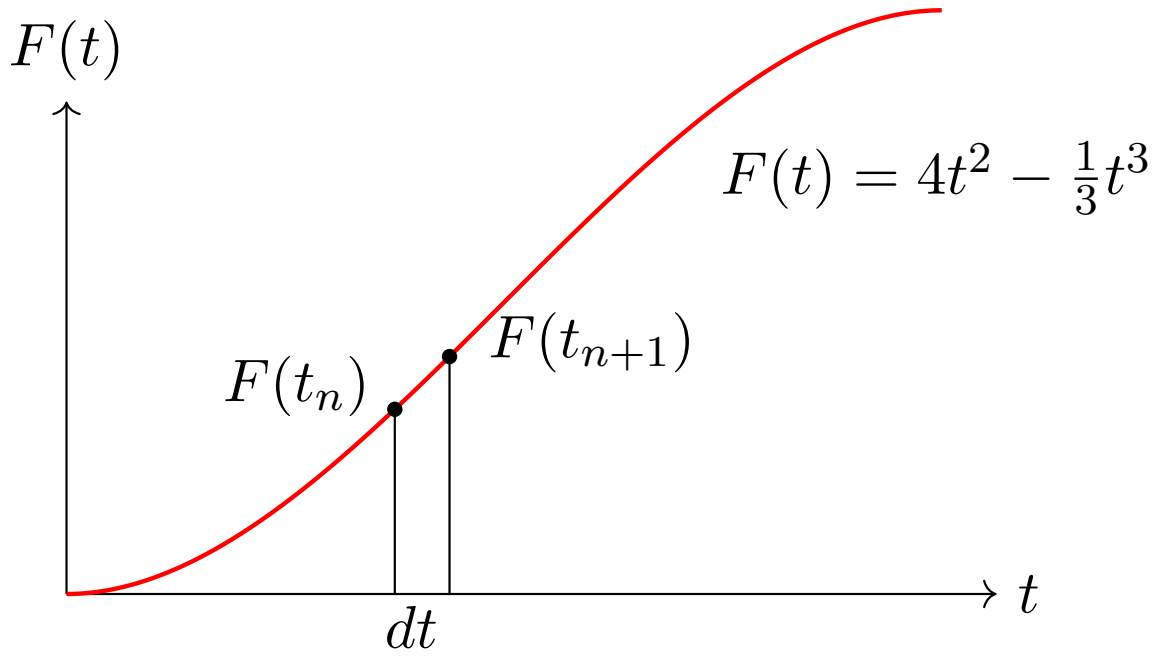
- Velocity: $v(t) = t(8 - t)$
- Expand: $8t - t^2$
- Antiderivative:

$$A(t) = \int (8t - t^2)\, dt = 4t^2 - \frac{1}{3}t^3 + C$$

Integration

- Then, recall for very small dt, we have

$$f(t_n) \approx \frac{F(t_{n+1}) - F(t_n)}{dt} \implies \underbrace{f(t_n)\, dt}_{dA} \approx F(t_{n+1}) - F(t_n)$$



Integration

- Then, the area A can be approximated by

$$A \approx \sum_{n=1}^{N-1} f(t_n)dt \approx \sum_{n=1}^{N-1} (F(t_{n+1}) - F(t_n))$$

$$= F(t_2) - F(t_1) \quad + \quad F(t_3) - F(t_2)$$
$$+ \quad F(t_4) - F(t_3) \quad + \cdots + \quad F(t_N) - F(t_{N-1})$$

$$= F(t_N) - F(t_1)$$
$$= F(b) - F(a)$$

Fundamental Theorem of Calculus

- In the limit, we write $\sum_{n=1}^{N-1} f(t_n)\,dt$ into $\int_a^b f(t)\,dt$, and have

$$A = \int_a^b f(x)\,dx = F(b) - F(a)$$

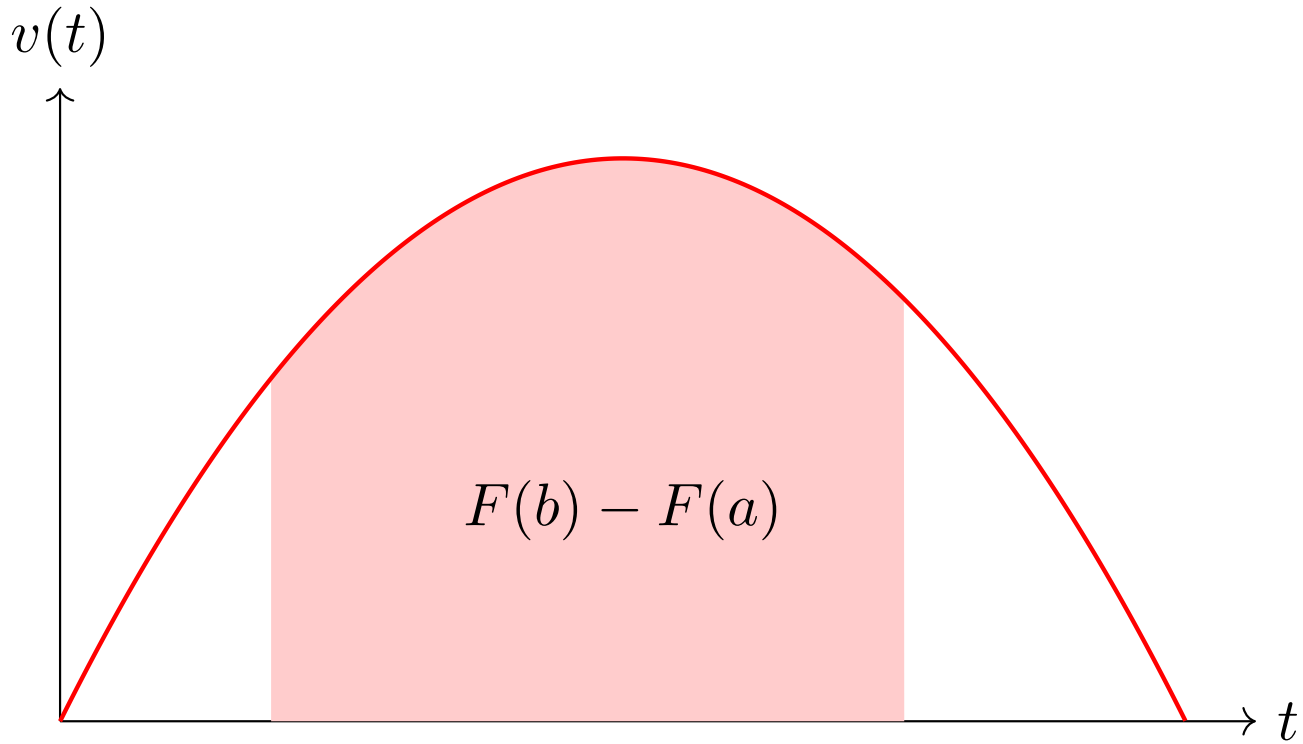
- The integral considers every value between a and b, but the result depends only on the endpoints!

- This links the algebraic world of derivatives with the geometric world of areas.

Fundamental Theorem of Calculus

- If $F'(x) = f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$



Antiderivative of Example

- Velocity: $v(t) = t(8 - t)$
- Antiderivative:

$$A(t) = \int_0^8 (8t - t^2)\, dt = \left[4t^2 - \frac{1}{3}t^3 + C\right]_0^8$$

- Distance between 0 and 8:

$$A = 4(8^2) - \frac{1}{3}(8^3) - 4(0^2) - \frac{1}{3}(0^3) = 85.333$$

- This equals the exact area under the velocity curve

Signed Area

- Integrals measure ‘signed area’, so negative velocity gives negative contribution.

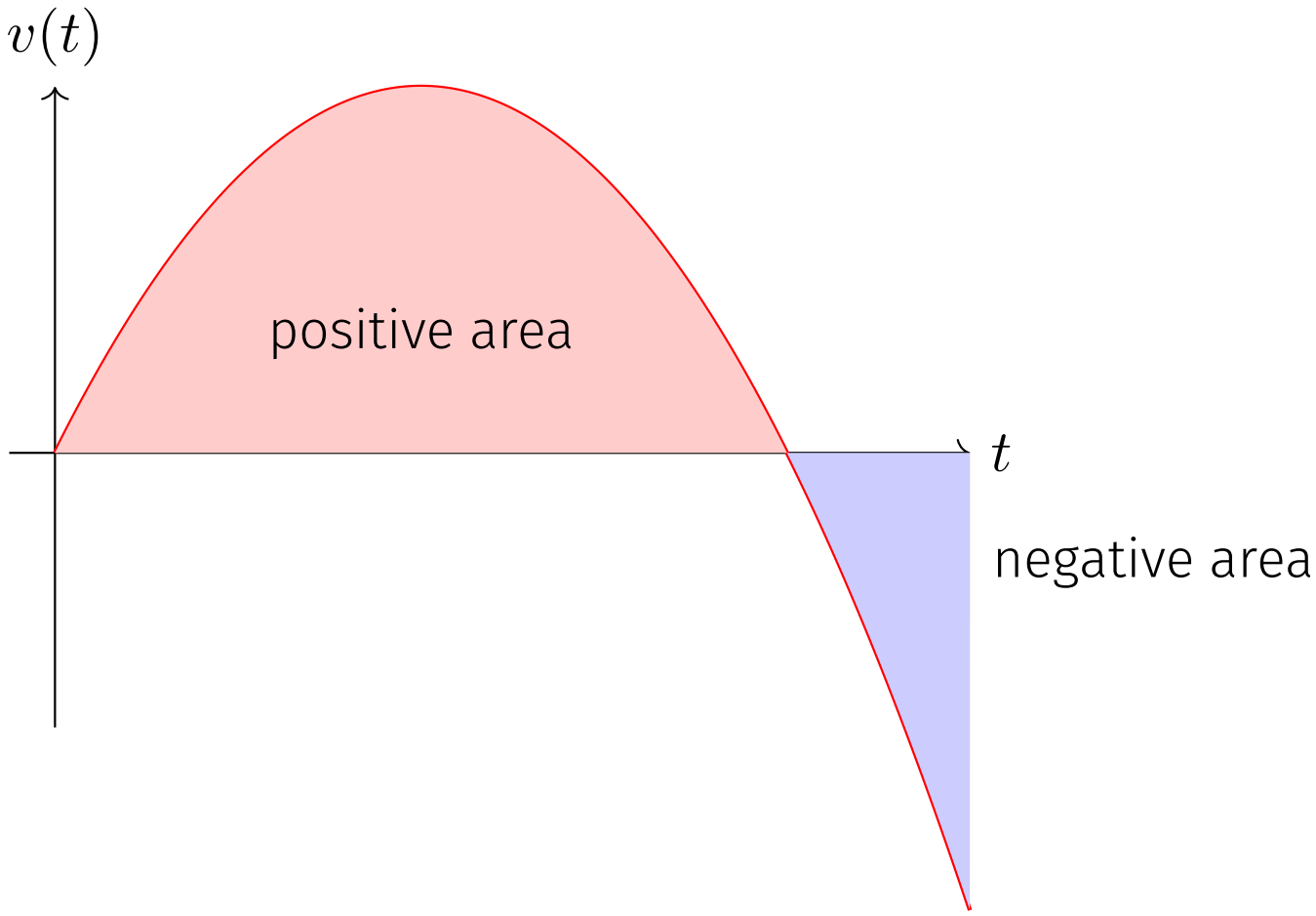


Table of integrals

- basic integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$x^{n+1}/(n+1)+C$	$\sin x$	$-\cos x+C$
$\ln x$	$x\ln x-x+C$	$\cos x$	$\sin x+C$
e^x	e^x+C	$\tan x$	$\ln\lvert\sec x\rvert+C$
a^x	$a^x/\ln a+C$	$\arcsin x$	$x\arcsin x+\sqrt{1-x^2}+C$
e^{kx}	$\frac{e^{kx}}{k}+C$	$\arccos x$	$x\arccos x-\sqrt{1-x^2}+C$
$\frac{1}{x}$	$\ln x+C$	$\arctan x$	$x\arctan x-1/2\ln\lvert1+x^2\rvert+C$

- where C is a constant
- look in the text book or in the web for more basic integrals

Functions of a linear function of x

- If

$$\int f(x)\,dx = F(x) + C$$

- then

$$\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + C$$

Functions of a linear function of x

- In general $\int f(ax+b)\,dx$, where we know $F'(x) = f(x)$ and let $z = (ax+b)$
- then $\frac{dz}{dx} = a$ or $dx = \frac{1}{a}\,dz$, consider

$$\begin{aligned}\int f(ax+b)\,dx &= \int f(z)\frac{1}{a}dz \\ &= \frac{1}{a}\int f(z)\,dz = \frac{1}{a}F(z) + C \\ &= \frac{1}{a}F(ax+b) + C\end{aligned}$$

Example: change of variables

- Find $\int (5x-4)^6 dx$
- make the change of variables from x to z where $z = 5x - 4$
- notice that $dz = 5dx$ or $dx = \frac{1}{5}dz$

$$\begin{aligned}\int (5x-4)^6\, dx &= \int z^6 \times \frac{1}{5} dz \\ &= \frac{1}{5}\int z^6\, dz\end{aligned}$$

- the integral $\int z^6\, dz$ is done using our basic formula $\int x^n dx = x^{n+1}/(n+1) + C$ with $n = 6$ gives

$$\frac{1}{5}\int z^6\, dz = \frac{1}{5} \times \frac{1}{7} z^7 + C = \frac{z^7}{35} + C = \frac{(5x-4)^7}{35} + C$$

Integrals of the form $f'(x)/f(x)$

- The integral of

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{1}{f(x)}d(f(x)) = \ln f(x) + C$$

- Derivation: Consider $\ln(f(x))$ and differentiate using the chain rule

$$\frac{d}{dx}\, g(f(x)) = g'(f(x))f'(x) \implies \frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)}f'(x)$$

- integrate

$$\begin{aligned} \int \frac{d}{dx}(\ln(f(x)))dx &= \int \frac{f'(x)}{f(x)}dx \\ \ln(f(x)) + C &= \int \frac{f'(x)}{f(x)}dx \end{aligned}$$

Example

- Find
$$\int \frac{2x+3}{x^2+3x-5}dx$$
- notice that $2x+3$ is the derivative of x^2+3x-5
- then applying the formula we just learned
$$\int \frac{2x+3}{x^2+3x-5}dx = \ln(x^2+3x-5)+C$$

Integrals of the form $f'(x) \times f(x)$

- The integral of

$$\int f'(x) \times f(x) dx = \int f(x) d(f(x)) = \frac{(f(x))^2}{2} + C$$

- Derivation. Differentiate using the chain rule $\frac{d}{dx}(f(x))^2 = 2f(x)f'(x)$ then $f(x)f'(x) = \frac{1}{2}\frac{d}{dx}(f(x))^2$
- Integrate

$$\begin{aligned} \int f'(x) \times f(x) dx &= \frac{1}{2} \int \frac{d}{dx} \left((f(x))^2 \right) dx \\ \int f'(x) \times f(x) dx &= \frac{f(x)^2}{2} + C \end{aligned}$$

Example

- Find
$$\int \tan x \, \sec^2 x \, dx$$
- Method 1: Notice that $\frac{d}{dx} \tan x = \sec^2 x$ then the integral is of the form $f(x)f'(x)$
- applying the formula
$\int f'(x) \times f(x)dx = \int f(x)df(x) = \frac{(f(x))^2}{2} + C$ then
$$\int \tan x \sec^2 x \, dx = \frac{1}{2} \tan^2 x + C$$

Example

- Find
$$\int \tan x \sec^2 x \, dx$$
- Method 2: Change of variables. Let $z = \tan x$ so $\frac{dz}{dx} = \sec^2 x$ then $dz = \sec^2 x \, dx$

$$\int \tan x \sec^2 x \, dx = \int z \, dz = \frac{z^2}{2} + C = \frac{1}{2} \tan^2 x + C$$

Integration by parts

- if u and v are functions of x, then by the chain rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- Integrate both sides with respect to x

$$\int \frac{d}{dx}(uv)\,dx = uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

- rearrange the terms

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx = uv - \int v du$$

Integration by parts

- From

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx = uv - \int v du$$

- simplify to

$$\int u\,dv \;\; = \;\; uv - \int v\,du$$

Example

- Find
$$\int xe^x\,dx$$
- let $u = x$ and $\frac{dv}{dx} = e^x$ so $\frac{du}{dx} = 1$ and $v = e^x$ (v obtained by integration)
- integrate by parts $\int u\,dv = uv - \int v du$

$$\int xe^x\,dx = xe^x - \int e^x\,dx + C = e^x(x-1) + C$$

Integration partial fractions

- If the integrand is an algebraic fraction that can be separated into its partial fractions then each individual partial fraction can be integrated separately.
- Example: Find $\int \frac{x+1}{x^2-3x+2}\,dx$
- We need to express $x^2 - 3x + 2$ as a partial fraction and then integrate by parts.

Integration partial fractions

- Factorize the denominator

$$\frac{x+1}{x^2-3x+2} = \frac{x+1}{(x-2)(x-1)}$$

- to decompose in partial fractions, find A and B such that

$$\frac{x+1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1}$$

- find $x+1 = A(x-1) + B(x-2)$ set $x=2$ then $A=3$ and set $x=1$ then $B=-2$

Integration partial fractions

- Example: Find $\int \frac{x+1}{x^2-3x+2}\,dx$
- Now we have the partial fraction we do the integration

$$\int \frac{x+1}{x^2-3x+2}\,dx = \int \frac{3}{x-2}dx - \int \frac{2}{x-1}dx$$

- if $u_1 = x-2$ and $u_2 = x-1$ then $\frac{du_1}{dx} = 1$ and $\frac{du_2}{dx} = 1$ and $du_1 = dx$ and $du_2 = dx$

$$\int \frac{x+1}{x^2-3x+2}\,dx = \int \frac{3}{x-2}dx - \int \frac{2}{x-1}dx = \int \frac{3}{u_1}du_1 - \int \frac{2}{u_2}du_2$$

- then

$$\int \frac{x+1}{x^2-3x+2}\,dx = 3\ln(x-2) - 2\ln(x-1) + C$$

Integration of trigonometric functions

- Many integrals with trigonometric integrands can be evaluated after applying trigonometric identities.
- Example

$$\begin{aligned}\int \sin^2 x\, dx &= \int \frac{1}{2}(1-\cos(2x))\, dx \\ &= \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\, dx \\ &= \frac{x}{2} - \frac{\sin(2x)}{4} + C\end{aligned}$$

Example

- Find

$$\int \sin^5 x \, dx$$

- rewrite

$$\begin{aligned} \sin^5 x &= \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x \\ &= (1 - 2\cos^2 x + \cos^4 x) \sin x \\ &= \sin x - 2\cos^2 x \sin x + \cos^4 x \sin x \end{aligned}$$

- now

$$\int \sin^5 x \, dx = \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x) \, dx$$

Example

- now if $u = \cos x$ then $\frac{du}{dx} = -\sin x$ and $du = \sin x \, du$

$$\begin{aligned}
\int \sin^5 x \, dx &= \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x) \, dx \\
&= \int \sin x \, dx - 2 \int \cos^2 x \sin x \, dx \\
&\quad + \int \cos^4 x \sin x \, dx \\
&= -\cos x + 2 \int u^2 \, du - \int u^4 \, du \\
&= -\cos x + \frac{2}{3} u^3 - \frac{1}{5} u^5 \\
&= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C
\end{aligned}$$

Example (Complex Numbers Method)

- We use Euler's formula:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- Then

$$\sin^5 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^5$$

Expand using the binomial theorem:

$$\sin^5 x = \frac{1}{32i^5}\left(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix}\right)$$

Since $i^5 = i$, this becomes:

$$\sin^5 x = \frac{1}{32i}\left(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix}\right)$$

Convert Back to Trigonometric Functions

Group conjugate pairs:

$$\sin^5 x = \frac{1}{16}\Big(\sin(5x) - 5\sin(3x) + 10\sin(x)\Big)$$

Thus we reduce it to easier integrals:

$$\int \sin^5 x\,dx = \frac{1}{16}\left(\int \sin 5x\,dx - 5\int \sin 3x\,dx + 10\int \sin x\,dx\right)$$

Therefore:

$$\int \sin^5 x\,dx = \frac{1}{16}\left(-\frac{1}{5}\cos 5x + \frac{5}{3}\cos 3x - 10\cos x\right) + C$$

Simplify:

$$\boxed{\int \sin^5 x\,dx = -\frac{1}{80}\cos 5x + \frac{5}{48}\cos 3x - \frac{5}{8}\cos x + C}$$

The End