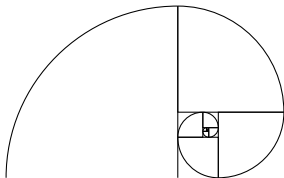


# Sequences



0,1,1,2,3,5,8,13,...

# Sequences

- A sequence is a function  $f$  whose input is restricted to the integers  $(\dots 0, 1, 2, \dots)$  and the output is an enumerated collection of numbers.

- Example

$$f(n) = \left(\frac{1}{2}\right)^n$$

- The input is the number  $n = 0, 1, 2, \dots$

we are only considering sequences whose output is numbers

# Arithmetic sequence

- ❖ The sequence

$$f(n) = a + nd, \quad n = 0, 1, 2 \dots$$

where  $a$  and  $d$  are numbers is an arithmetic sequence

- ❖ Each term is obtained by adding  $d$  to the previous term
- ❖  $d$  is called the common difference
- ❖ Example

$$f(n) = 2 + n \times 3, \quad n = 0, 1, 2 \dots$$

- ❖ then  $f(0) = 2$ ,  $f(1) = 2 + 1 \times 3 = 5$ ,  
 $f(2) = 2 + 2 \times 3 = 8$

# Arithmetic sequence

- ❖ Properties of the sequence

$$f(n) = a + nd, \quad n = 0, 1, 2 \dots$$

- ❖ take the difference of two consecutive elements of the sequence, that is

$$f(n+1) - f(n) = (a + (n+1)d) - (a + nd) = a + nd + d - a - nd = d$$

- ❖ the difference of consecutive terms gives  $d$
- ❖ Now consider the term  $f(0)$

$$f(0) = a + 0 \times d = a$$

- ❖ the value of  $a$  is given by  $f(0)$ .

# Arithmetic sequence

- Example Find the function that describes the sequence

$$-3, 4, 11, 18 \dots$$

- Check the difference between an element of the sequence and the next element, that is  $4 - (-3) = 7$ ,  $11 - 4 = 7$ ,  $18 - 11 = 7$ 
  - as the difference is a constant value 7 then is a geometric series and the common difference  $d = 7$
- As  $f(n) = a + 7 \times n$  the value of  $a$  is obtained from  $f(0)$  which is  $f(0) = a = -3$ ,
- the series is

$$f(n) = -3 + 7n, \quad n = 0, 1, 2, 3, \dots$$

# Geometric sequence

- ❖ The sequence

$$f(n) = A r^n, \quad n = 0, 1, 2 \dots$$

where  $A$  and  $r$  are numbers is a geometric sequence

- ❖ Each term is obtaining by multiplying by  $r$  to the previous term
- ❖  $r$  is called the common ratio
- ❖ Example

$$f(n) = 3 \times 2^n, \quad n = 0, 1, 2 \dots$$

- ❖ then  $f(0) = 3 \times 2^0 = 3$ ,  $f(1) = 3 \times 2^1 = 6$ ,  
 $f(2) = 3 \times 2^2 = 12$

# Geometric sequence

- ❖ Properties of the sequence

$$f(n) = A r^n, \quad n = 0, 1, 2 \dots$$

- ❖ take the ratio of two consecutive elements, that is

$$\frac{f(n+1)}{f(n)} = \frac{A r^{n+1}}{A r^n} = \frac{r A r^n}{A r^n} = r$$

- ❖ the ratio give us the value of  $r$
- ❖ Take the first element of the series  
 $f(0) = A r^0 = A \times 1 = A.$

# Example

- Find the function of the geometric sequence

$$4, 12, 36, 108, \dots$$

- To get the common ratio take the ratio of two consecutive terms. Then  $12/4 = 3$ ,  $36/12 = 3$ ,  $108/36 = 3$  then  $r = 3$
- To get  $A$  evaluate  $f(0)$  which is  $f(0) = A \times 3^0 = A \times 1 = 4$ , then  $A = 4$

# Harmonic sequence

- ❖ The sequence

$$f(n) = \frac{1}{n} \quad n = 1, 2, \dots$$

- ❖ is called a harmonic sequence. Is the reciprocal form of the arithmetic sequence  $1, 2, \dots$
- ❖ Example. If the arithmetic sequence is  $4, 6, 8, 10, \dots$  then the harmonic sequence is

$$\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots$$

# Harmonic sequence

- ❖ Generalising, if  $f(n) = a + d \times n$  with  $n = 0, 1, \dots$  then the harmonic sequence is

$$g(n) = \frac{1}{f(n)} = \frac{1}{a + d \times n}, \quad n = 0, 1, \dots$$

- ❖ Harmonic sequences are relevant in electrical gadgets and generation power
- ❖ The value  $a + d \times n$  should be different from zero.

# Describing Sequences

- There is more than one way to describe a sequence

- Direct prescriptions

$$f(n) = a + nd, \quad n = 0, 1, 2, \dots$$

- where the values of  $a$  and  $d$  are given
- Recursive prescriptions

$$f(n + 1) = f(n) + d, \quad \text{and } f(0) = a$$

- Notice the condition  $f(0) = a$  is called the **initial condition** and is needed to describe the sequence
- this initial condition gives the value of the sequence for an specific  $n$

# Example

- Direct prescription

$$f(n) = -2 + 3n, \quad n = 0, 1, 2, \dots$$

gives  $f(0) = -2$ ,  $f(1) = 1$ ,  $f(2) = 4$

- Recursive description

- evaluate  $f(n+1)$  and write it in terms of  $f(n)$

$$f(n+1) = -2 + 3(n+1) = -2 + 3n + 3 = f(n) + 3$$

- then  $f(n+1) = f(n) + 3$ . Suppose that  $f(2) = 4$  then the previous value  $f(1)$  is obtained from

$$f(2) = f(1) + 3 = 4$$

with solution  $f(1) = 1$ , then  $f(1) = f(0) + 3 = 1$  then  $f(0) = -2$  and  $f(3) = f(2) + 3 = 7$

# Difference equations

- ❖ The recursive form

$$f(n+1) = f(n) + 5$$

- ❖ can be written as

$$f(n+1) - f(n) = 5$$

- ❖ this is an example of a difference equation  
 $f(n+1) - f(n) = d$

# Difference equations

- ❖ The difference equation

$$f(n+1) - f(n) = d$$

- ❖ is an example of a first-order, constant coefficient, linear difference equation
  - ❖ linear because there are no products of terms such as  $f(n+1) \times f(n)$
  - ❖ first-order because  $f(n+1)$  is just one term away from  $f(n)$ ,
  - ❖ it has constant coefficients (the numbers multiplying the  $f(n)$  and  $f(n+1)$  are constants and do not involve  $n$ )

# Difference equations

- ❖ The order of equation is defined by the maximum difference between pair of sequence elements.

Example

$$f(n+2) + 2f(n) = 3n^4 + 2$$

- ❖ is a second-order difference equation because  $f(n+2)$  is two terms away from  $f(n)$ .
- ❖ **Initial conditions** In order to generate the terms of the sequence from the recursive description it is necessary to have as many initial terms (initial conditions) as the order of the difference equation.
- ❖ In the above example we need two initial terms to describe the sequence fully.

# Difference equations

- ❖ How to find from a difference equation the sequence?
- ❖ Example if we have

$$f(n+1) - f(n) = 5$$

- ❖ what is  $f(n)$  explicitly that is

$$f(n) = 5n - 2, \quad n \geq 1$$

- ❖ This is called solving the difference equation.
- ❖ For some difference equation it is possible to find the sequence

## Homogenous First Order difference equations

- Consider the difference equation of the form

$$a f(n+1) + b f(n) = 0$$

- is a difference equation of first order
- is homogenous because the right-hand side is zero
- Example

$$f(n+1) + 9f(n) = 0, \quad n \geq 0, \text{ and } f(0) = 6$$

- is a first order homogenous difference equation with initial condition  $f(0) = 6$

## Solving Homogenous First Order difference equations

- Assume that the solution is of the form

$$f(n) = kw^n$$

- where  $k$  and  $w$  are non-zero real numbers and  $n$  is an integer
- Example solve

$$f(n+1) + 9f(n) = 0, \quad n \geq 0, \text{ and } f(0) = 6$$

- using  $f(n) = kw^n$  gives (**the characteristic equation**)

$$\begin{aligned} kw^{n+1} + 9kw^n &= 0 \\ w^n k (w + 9) &= 0 \end{aligned}$$

## Solving Homogenous First Order difference equations

- ❖ The characteristic equation is  $w^n k (w + 9) = 0$
- ❖ a solution is  $w = -9$ . We are not interested in the solutions  $k = 0$  or  $w = 0$ . Then we can write

$$f(n) = k(-9)^n$$

- ❖ as  $f(0) = 6$  then  $f(0) = k(-9)^0 = 6$  or  $\boxed{k = 6}$  so finally

$$f(n) = 6 \times (-9)^n, \quad n \geq 0$$

# Example

▣ Solve

$$f(n+1) + 8f(n) = 0, \quad n \geq 0$$

▣ with  $f(0) = 5$ .

## Solve second order homogenous difference equation

- ❖ The equations are of the form

$$af(n+2) + bf(n+1) + cf(n) = 0$$

- ❖ same procedure as first order homogenous equations, that is, use  $f(n) = kw^n$
- ❖ but the characteristic equation has two roots and so a linear combination of two solutions is required
- ❖ and two initial conditions

## Solve second order homogenous difference equation

- Example, solve

$$f(n+2) - 7f(n+1) + 12f(n) = 0, \quad n \geq 0$$

and  $f(0) = 0$  and  $f(1) = 1$

- the characteristic equation (use  $f(n) = kw^n$ )

$$kw^{n+2} - 7kw^{n+1} + 12kw^n = 0$$

$$kw^n (w^2 - 7w + 12) = 0$$

$$w^2 - 7w + 12 = 0$$

$$(w - 3)(w - 4) = 0$$

- we have two solutions  $w = 3$  and  $w = 4$

# Example

- Both,  $f_1(n) = k3^n$  and  $f_2 = k4^n$  are solutions of  $f(n+2) - 7f(n+1) + 12f(n) = 0$
- To include this two solutions the general solution is of the form

$$f(n) = Af_1(n) + Bf_2(n)$$

- where  $A$  and  $B$  are values that we need to find, to do so we use the initial conditions ( $f(0) = 0$  and  $f(1) = 1$ )

$$f(0) = A + B = 0 \quad (1)$$

$$f(1) = 3A + 4B = 1 \quad (2)$$

- Solving (1) and (2) gives  $A = -1$  and  $B = 1$   
**what happen to  $k$ ?**

# Example

- The solution is

$$f(n) = -3^n + 4^n, \quad n = 0, 1$$

# Characteristic equation with equal roots

- ❖ If the roots of the characteristic equation are equal then a different form for the solution must be given.
- ❖ Example

$$f(n+2) - 4f(n+1) + 4f(n) = 0 \quad n \geq 0$$

- ❖ has the characteristic equation

$$kw^n (w-2)^2 = 0$$

- ❖  $(w-2)^2 = (w-2)(w-2) = 0$  so we have two repeated solutions.
- ❖ To obtain the general solution use instead

$$f(n) = (A + Bn)w^n$$

- ❖ in the example  $f(n) = (A + Bn)2^n$
- ❖ to find  $A$  and  $B$  we use the initial conditions as previously

# Inhomogeneous Difference equations

❖ If

$$f(n+1) + af(n) = b$$

- ❖ where  $a$  and  $b \neq 0$  are constants.
- ❖ This is an inhomogeneous equation as we have a non-zero term on the right hand side of the equation
- ❖ To solve this kind of equations
  - ❖ Solve the homogenous equation, i.e. assume  $b = 0$ , the solution is  $f_H(n)$
  - ❖ Find a particular solution for the inhomogeneous equation, the solution is  $f_I(n)$
  - ❖ The general solution is the addition of these two solutions  $f(n) = f_H(n) + f_I(n)$

Example:  $f(n+1) + 8f(n) = 2$ ,  $n \geq 0$  and  $f(0) = 5$

- ❖ Solve the homogenous equation  $f(n+1) + 8f(n) = 0$ 
  - ❖ the characteristic equation is

$$kw^{n+1} + 8kw^n = 0$$

- ❖ the solution is  $w = -8$ ,  $f_H(n) = k(-8)^n$
  - ❖ for the inhomogeneous equation, propose that the solution is  $f(n) = c$ , then

$$c + 8c = 2$$

- ❖ with solution  $c = 2/9$ .
  - ❖ add the solutions  $f(n) = k(-8)^n + 2/9$  and apply the initial conditions,  $f(0) = 5$  which gives  
 $f(0) = k + 2/9 = 5$  or  $k = 5 - 2/9$

- ❖ finally

$$f(n) = (5 - 2/9)(-8)^n + 2/9, \quad n \geq 0$$

# Limits of sequences

- ❖ The number that the output of a sequence approaches as the input increases without bound is called the limit of the sequence.
- ❖ example,  $f(n) = 1/n$  no matter how large  $n$  becomes  $f(n)$  never attains the value of 0.
- ❖ but  $f(n)$  can become as close to 0 as we wish by choosing  $n$  to be sufficiently large.
- ❖ Zero “0” is the limit of the sequence as  $n$  approaches infinity ( $n \rightarrow \infty$ ) and write

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0$$

# Infinite limits

- ✦ Sometimes as  $n$  becomes large so does  $f(n)$ .
- ✦ Example:  $f(n) = n^3$ , in this case

$$\lim_{n \rightarrow \infty} (n^3) = \infty$$

- ✦ Notice that this notation does not mean that the limit is equal to infinity. It cannot be equal to infinity because infinity is not numerically defined so nothing can be said to be equal to it.
- ✦ What it means is that as  $n$  becomes large without bound then so does  $n^3$ .

## A note on $\lim_{n \rightarrow \infty} f(n) = \infty$

❖ Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$  then  $\lim_{n \rightarrow \infty} f(n) = \infty$  means

❖ For every  $\mathcal{M} \in \mathbb{R}$ , there exists a number  $\mathcal{N} \in \mathbb{R}$  with property that  $f(x) > \mathcal{M}$  for all values of  $x > \mathcal{N}$ .

Important result:

$$\lim_{n \rightarrow \infty} k^n = \begin{cases} 0, & -1 < k < 1 \\ 1, & k = 1 \\ \infty, & k > 1 \\ \text{undefined}, & k \leq -1 \end{cases}$$

❖ Example  $\lim_{n \rightarrow \infty} (-3)^n$  is undefined.

$$\lim_{n \rightarrow \infty} 4^n = \infty$$

$$\lim_{n \rightarrow \infty} (1/2)^n = 0$$

# Rules of limits

- ❖ Multiplication by a constant
- ❖ The limit of an expression multiplied by a constant is the constant multiplying the limit of the expression:

$$\lim_{n \rightarrow \infty} k f(n) = k \lim_{n \rightarrow \infty} f(n)$$

where  $k$  is a constant.

- ❖ The limit of a sum (or difference) is the sum (or difference) of the limits:

$$\lim_{n \rightarrow \infty} (f(n) \pm g(n)) = \lim_{n \rightarrow \infty} f(n) \pm \lim_{n \rightarrow \infty} g(n)$$

# Rules of limits

- ❖ Products and quotients
- ❖ The limit of a product (or quotient) is the product (or quotient) of the limits:

$$\lim_{n \rightarrow \infty} (f(n) \times g(n)) = \lim_{n \rightarrow \infty} f(n) \times \lim_{n \rightarrow \infty} g(n)$$

- ❖ and

$$\lim_{n \rightarrow \infty} \left( \frac{f(n)}{g(n)} \right) = \frac{\lim_{n \rightarrow \infty} f(n)}{\lim_{n \rightarrow \infty} g(n)}$$

provided that  $\lim_{n \rightarrow \infty} g(n) \neq 0$

# Indeterminate limits

- ✦ Sometimes when trying to determine the limit of a quotient the limits of both the numerator and the denominator are infinite.
- ✦ Such a limit is called an **indeterminate limit**
- ✦ and cannot be found without some manipulation on the quotient.

# Example

- ❖ Evaluate

$$\lim_{n \rightarrow \infty} \left( \frac{n^2 + 3}{4n^2 + 6} \right)$$

- ❖ notice that the limits of the numerator and denominator tend to infinity,  
 $\lim_{n \rightarrow \infty} n^2 + 3 = \infty$  and  $\lim_{n \rightarrow \infty} 4n^2 + 6 = \infty$
- ❖ trick to find the limit, divide the top and bottom by  $n^2$ , that is

$$\left( \frac{n^2 + 3}{4n^2 + 6} \right) = \left( \frac{n^2 + 3}{4n^2 + 6} \right) \left( \frac{n^2}{n^2} \right) = \frac{1 + 3/n^2}{4 + 6/n^2}$$

# Example

❖ Then

$$\lim_{n \rightarrow \infty} \left( \frac{n^2 + 3}{4n^2 + 6} \right) = \lim_{n \rightarrow \infty} \left( \frac{1 + 3/n^2}{4 + 6/n^2} \right)$$

❖ but now we have that

$$\lim_{n \rightarrow \infty} 1 + 3/n = 1 \text{ and}$$

$$\lim_{n \rightarrow \infty} 4 + 6/n^2 = 4 \text{ are finite}$$

❖ then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \frac{n^2 + 3}{4n^2 + 6} \right) &= \frac{\lim_{n \rightarrow \infty} (1 + 3/n^2)}{\lim_{n \rightarrow \infty} (4 + 6/n^2)} \\ &= \frac{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} (3/n^2)}{\lim_{n \rightarrow \infty} 4 + \lim_{n \rightarrow \infty} (6/n^2)} = \frac{1}{4} \end{aligned}$$

# Limiting values

- An *indeterminate form* in a limit problem is one where a limit of a ratio is to be determined where both the numerator and the denominator in the ratio have either a zero limit or an infinite limit.
- That is, a problem to determine:

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$$

- where  $\lim_{n \rightarrow \infty} f(n) \rightarrow 0$  and  $\lim_{n \rightarrow \infty} g(n) \rightarrow 0$ , or
- where  $\lim_{n \rightarrow \infty} f(n) \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} g(n) \rightarrow \infty$