Advanced Control Systems

Lecture 9: Filtering and State Observers

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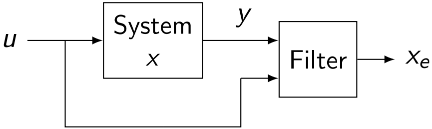
ECS654U/ECS778P

Introduction

Filtering is the problem of constructing a dynamical system to estimate the state of another system.

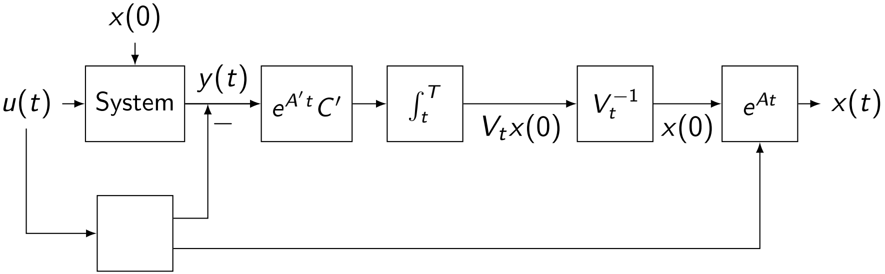


The filter takes available information (input, u and output, y) to generate an estimate x_e.

The goal is to minimize the estimation error $e = x - x_e$.

Challenges of Direct Reconstruction

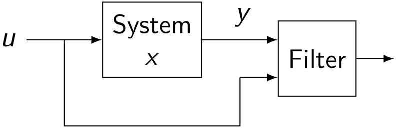
We saw in a previous lecture a direct way to get the state of the system if the system is observable:



However, we also mentioned that there are many issues with this approach, and it will often fail due to the accumulation of errors.

An Asymptotically Online Estimate

When using a filter, we want the estimation error e to converge to zero over time.



x_e, estimate of the state, x, of the system

That is to say

$$\lim_{t \to \infty} \underbrace{x - x_e}_{\substack{\text{e} \\ \text{estimation error}}} = 0$$

Notice that this is a type of stability requirement.

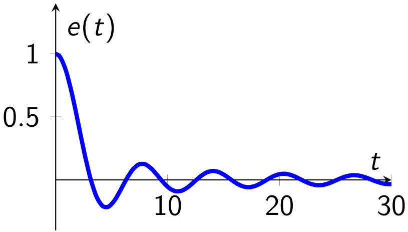
The question is, how do we connect the notion of an asymptotically online estimate (of the internal state of the system using another device) as a stability property?

Estimation Error

It should be mentioned that for $t = 0$ or $k = 0$, the estimation error, e, will be non-zero. That is to say

$$e(t)\Big|_{t=0} \neq 0$$

A typical behaviour of $e(t)$ could be something like this:



Recall that linear systems' convergence and asymptotic stability are the same property.

Comparison With Other Disciplines

- Signal processing also uses (digital) filters but with no underlying model, only measurements.
  - Example: Estimating object position from radar signals follows the same structure.
    - No assumption about the object's mechanical properties.
    - Only using signal measurements to infer position.

- Similar to architectures are used in image processing and communication.

- In control theory, we assume the existence of a model and this model is exploited in designing the filter. However, it is important to note that the overall architecture remains similar.

Filter Design Equations: Static Filters

Recall that we are studying linear systems described by the system equations

$$\sigma x = Ax + Bu$$
$$y = Cx \qquad (D = 0 \text{ for simplicity})$$

In terms of the filter design, we have two options:

- **Static Filter:**

  $$x_e = My + Nu$$

  Selecting M and N such that x_e is a 'good' estimate.

  This approach often fails, which we can see in the SISO case because we are trying to estimate n states with only two signals, u and y.

  In general, we simply lack enough information to solve the problem.

Filter Design Equations: Dynamic Filters

To solve this problem with static filters, we need to be able to store information on u and y inside the filter; for this, we need a dynamic filter containing a memory.

- **Dynamic Filter** (equivalent to the static case when $\dim \xi = 0$):

$$\sigma\xi = F\xi + Ly + Hu$$
$$x_e = M\xi + Ny + Pu$$

Therefore, a dynamic filter can address this dimensionality limitation by storing and processing past data over time.

Notice that the design parameters for the dynamic filter are the $\dim \xi$ along with the matrices F, L, H, M, N and P.

However, on the face of it, this looks like overkill because now we have seven design parameters.

Observer-Based Filtering

So, is there a natural way to find the matrices F, L, H, M, N and P, which will give us an asymptotically online (continuously improving) estimate of the system?

$$\sigma\xi = F\xi + Ly + Hu$$
$$x_e = M\xi + Ny + Pu$$

First, we would like to simplify the second equation so that it has no design parameters (and all past information is stored in ξ).

To achieve this we set $M = I$, $N = 0$, $P = 0$ which simplifies equation to

$$x_e = \xi \,, \qquad \text{(where we would like } x_e \approx x \text{ so the } \dim \xi = n\text{)}$$

This leads to a sort of identity filter where the state ξ is a copy of x_e

This special case where x_e approximates x directly is often referred to as a **state observer** (rather than a filter).

Dynamical Systems

So let us consider the two coupled dynamical systems

$$\sigma x = Ax + Bu \ , \qquad y = Cx$$
$$\sigma \xi = F\xi + Ly + Hu \ , \qquad x_e = \xi$$

Our goal is to find the design parameters, which are matrices F, L, and H.

We start by recalling the definition of the estimation error

$$e = x - x_e = x - \xi$$

We seek two key properties:

- If $e(0) = 0$, then $e(t) = 0$ for all t (Consistency of Estimate).
- If $e(0) \neq 0$, then $\lim_{t\to\infty} e(t) = 0$ (Asymptotic Stability).

Both these properties have to be uniform with respect to u
(i.e. they have to be true no matter what u is).

Eliminating Dependence on u

From the error equation

$$\sigma e = Ax + Bu - (F\xi + Ly + Hu)$$

we obtain

$$\sigma e = (A - LC)x - F\xi + (B - H)u \qquad \text{(where } y = Cx\text{)}$$

Now, if we want to ensure independence from u, we must set

$$H = B$$

where B is given and H is a design parameter.

Invariance of the Error

So now we have the equation

$$\sigma e = (A - LC)x - F\xi$$

but it has three variables, which is not very useful. We would like to express this equation as a function of e.

$$e = x - \xi\ , \qquad \text{therefore,} \qquad x = e + \xi$$

which gives us

$$\sigma e = (A - LC)(e + \xi) - F\xi$$
$$\sigma e = (A - LC)e + (A - LC - F)\xi$$

Now recall that we require, if $e(0) = 0$, then $e(t) = 0$ for all t, leading to

$$\sigma e = (A - LC - F)\xi \qquad (\text{when } e = 0)$$

Therefore, to enforce invariance of the error, we set:

$$A - LC - F = 0 \Rightarrow F = A - LC.$$

Ensuring Asymptotic Stability

So, now that we have found equations for H and F, we need to work out the design parameter L using the equation.

$$\sigma e = (A - LC)e$$

Now recall that we require, if $e(0) \neq 0$, then $\lim_{t \to \infty} e(t) = 0$ (Asymptotic Stability).

This means the $A - LC$ needs to be an asymptotically stable matrix, i.e.

- **Continuous-time**: Eigenvalues of $A - LC$ all lie in $\mathbb{C}_{\text{good}}$ $(\mathbb{C}^{-})$
- **Discrete-time**: Eigenvalues of $A - LC$ all lie in $\mathbb{C}_{\text{good}}$ $(|\lambda_i| < 1 \forall i)$

Therefore, L should be assigned with a **asymptotic stability constraint**.

So, we now have a way of designing an observer using conditions on the matrices F, L, and H.

Designing Parameter L

The question remaining is how we should design L such that $\sigma e = (A - LC)e$ is asymptotically stable.

This is actually the same problem that we had when designing state feedback, and we wanted to assign the eigenvalues of $A + BK$.

We said that

$$(A, B) \text{ reachable} \quad \Longleftrightarrow \quad \begin{array}{c}\text{The eigenvalues of } A{+}BK \\ \text{can be arbitrarily assigned}\end{array}$$

where we need to assign complex conjugate pairs for K to be real valued.

The problem we face is that $A - LC$ and $A + BK$ are slightly different in terms of the order of the multiplication (note that the '$-$' sign is irrelevant).

The design parameter is on the right for BK and on the left for LC.

So how do we find a condition on (A, C)

Designing Parameter L via Duality

However, we know that we can go from reachability properties to observability properties using duality.

So if I take the transpose of $A - LC$, we get

$$(A - LC)' = A' - C'L'$$

which moves our design parameter, L, to the right. Therefore, we can say

$$(A', C') \text{ reachable} \iff \begin{array}{c}\text{The eigenvalues of } A' - C'L' \\ \text{can be arbitrarily assigned}\end{array}$$

which is equivalent to the observability of (A, C)

$$(A, C) \text{ observable} \iff \begin{array}{c}\text{The eigenvalues of } A - LC \\ \text{can be arbitrarily assigned}\end{array}$$

This gives us a conceptual way to assign the eigenvalues of $A - LC$.

Detectability and Observability

However, as we said for state feedback, observability of (A, C) is not always necessary to achieve the asymptotic stability constraint on e.

All we need is for all the unobservable modes to be in the good stability region. (Exactly the same as with stabilization using state feedback)

This is because we cannot move the unobservable modes using L, so they need to be asymptotically stable.

Therefore, we define a weaker property than observability, 'detectability'

(A, C) detectable $\iff$ There exists an L such that this system is asymptotically stable

which is equivalent to

(A, C) detectable $\iff$ All unobservable modes are in $\mathbb{C}_{\text{good}}$ (i.e. $\mathbb{C}^-$ or $|\lambda_i| < 1 \forall i$)

Notice that observability $\implies$ detectability *(the reverse is not true)*.

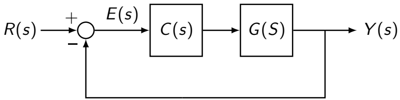
State Feedback and Observer Design

So, we now have two tools at our disposal:

- State feedback (from previous lecture).
- Observer design.

The goal is how we now use state feedback and an observer to design a stabilizing system controller for a dynamic system.

The traditional approach *(seen in the 'Control Systems' module)*: Design $C(s)$ such that the closed-loop system is asymptotically stable.



- A trial and error method:
  - Add integrators for performance.
  - Add zeros for phase margin.
  - Root locus for stability analysis.

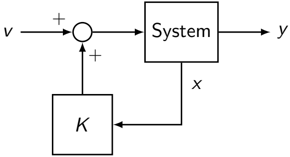
It works for simple SISO systems in continuous time but is not suitable for discrete-time or MIMO systems.

Observer-Based Control

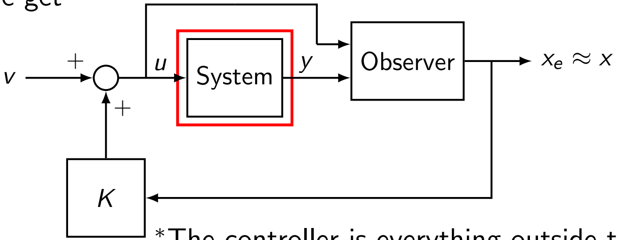
We want a systematic way to design a stabilizing system controller.

Consider the state-space system $\sigma x = Ax + Bu, \quad y = Cx$.

- If the state x is available, we apply state feedback $u = Kx$.
- However, in practice, we estimate the state using an observer.



Therefore we get



*The controller is everything outside the red box.

System Representation

So we have this state-space representation of the system

$$\sigma x = Ax + Bu$$
$$y = Cx$$

and the state-space representation of the observer

$$u = Kx = K\xi \quad \text{(where } \sigma x = (A + BK)x \text{ is asymptotically stable)}$$
$$\sigma\xi = (A - LC)\xi + LCx + BK\xi \quad \text{(where } F = A - LC, H = B, y = Cx, \text{ and } \sigma e = (A - LC)e \text{ is asymptotically stable)}$$

Therefore, the matrix of the overall closed loop system is

$$\begin{bmatrix} \sigma x \\ \sigma\xi \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} A & BK \\ \hline LC & A - LC + BK \end{array}\right]}_{A_{\text{cl}}} \begin{bmatrix} x \\ \xi \end{bmatrix}$$

However, it is disappointing that neither $A - LC$ nor $A + BK$ appear to show up in A_{cl}.

Coordinate Transformation

So what can we do? Well, if we write out the full equations, we get

$$\begin{aligned}\sigma x &= Ax + BK\xi \\ \sigma\xi &= (A - LC + BK)\xi + LCx\end{aligned}$$

and we want to relate this to $A - LC$ and $A + BK$.

However, to do this, we just need to write the equations in proper coordinates. We want to go from

$$\begin{bmatrix} x \\ \xi \end{bmatrix} \rightarrow \begin{bmatrix} x \\ e \end{bmatrix}$$

where $e = x - \xi$. Therefore we get

$$\begin{aligned}\sigma x &= Ax + BK(x - e) = (A + BK)x - BKe \\ \sigma e &= (A - LC)e\end{aligned}$$

Closed-Loop System

If we now write the following closed-loop system equations

$$\sigma x = Ax + BK(x - e) = (A + BK)x - BKe$$
$$\sigma e = (A - LC)e$$

we get

$$\begin{bmatrix} \sigma x \\ \sigma e \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} A + BK & -BK \\ \hline 0 & A - LC \end{array}\right]}_{\tilde{A}_{\text{cl}}} \begin{bmatrix} x \\ e \end{bmatrix}$$

So we have now identified exactly the two matrices $A - LC$ or $A + BK$ that we have designed:

- $A + BK$ (state feedback design)
- $A - LC$ (observer design)

Note that the matrix L is often called the output injection matrix.

Separation Principle

- State feedback and observer design can be treated separately.
- The closed-loop eigenvalues are determined by the individual eigenvalues of $A + BK$ and $A - LC$.
- Notice that observer-based controllers achieve stabilization with a small transient error, 'BKe', in

$$\sigma x = (A + BK)x - BKe$$

However, this term decays exponentially as e decays exponentially.
- Notice also that the system is not reachable

$$\left[\begin{array}{c|c} A + BK & -BK \\ \hline 0 & A - LC \end{array}\right]$$

In fact, by the PBH test, we note that the unreachable modes are all the eigenvalues of $A + LC$. *(Implying that the state observer does not contribute to the input-output behaviour of the closed-loop system.)*

Next Steps

This lecture concludes the theoretical part of the course.

The ‘module notes’ also include two more topics which we will not cover: reduced order observers and regulators.

Next week, we will go through some exercises.

Coursework 2 has been released and is on QMPlus (Due: Tuesday, 15 April 2025, 5:00 PM)