Advanced Control Systems

Lecture 8: Notion of Feedback

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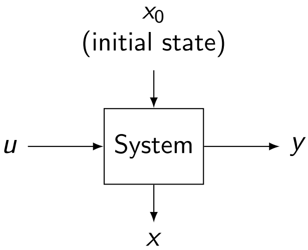
Introduction to Feedback

In this lecture, we are going to explore how we can modify a system's behaviour by applying feedback.

Given the state space equation for a linear system

$$\sigma x = Ax + Bu \,, \quad y = Cx$$

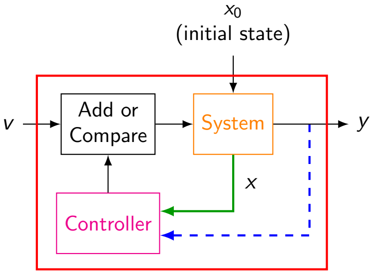
- Imagine we have access to the state of the system (an auxiliary measured output).
- Examples:
  - Mechanical systems: positions and velocities.
  - Electrical networks: currents and voltages.



Concept of Feedback

Feedback takes measured signals *(either state variables, x, or output variables, y)* and modifies the input to the **system**.

- **State feedback:** directly use x to modify input.
- **Output feedback:** use output y to modify input.
- Leads to a two-input configuration (the feedback signal and an external signal, v) where the **'Controller'** is a design parameter.



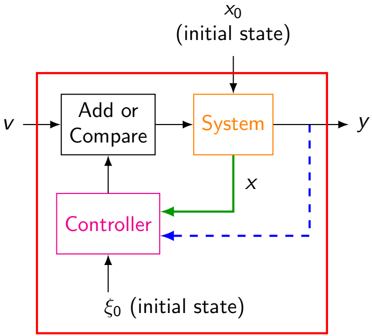
Notice that the **red box** is effectively just a new system with different behaviour where the input is v and the output is y.

Closed-Loop System

So it is clear to see that a **closed-loop system** can modify the **original system**.

- Notice that the **Controller** itself can be a dynamical system with internal state ξ_0.
- A linear **Controller** will allow us to design a controller choosing appropriate matrices.

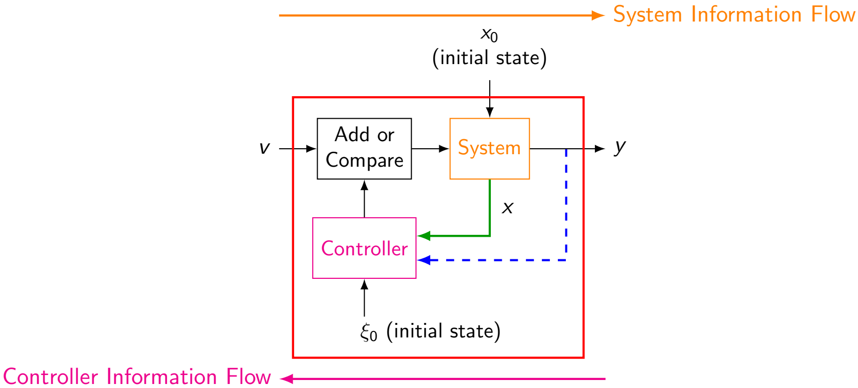
  Common designs in classical control *(seen in the 'Control Systems' module)* are:
  - Proportional Gain
  - Proportional Integral (PI) Controller



Flow of Information

If we look at the flow of information, we can see that

- System processes inputs to generate outputs.
- Controller operates in reverse, using outputs to generate control signals.
- Counter-flow of information forms a feedback loop.



Static Output Feedback

So, essentially, there are four different combinations that we can choose

- whether we use state or output feedback
- whether the controller is just an amplifier (just a scalar gain) or whether it is a dynamical system

Let's start with the simplest case, which is called **static output feedback** for system

$$\sigma x = Ax + Bu \ , \quad y = Cx$$

In this case, we modify system input in the following way

$$u = Ky + v$$

Notice that here, the input is y, and the output is u (i.e. reversed) with a new external input v.

Static Output Feedback: Closed-loop system equations

Therefore, the closed-loop system equations are

$$\begin{aligned}
\sigma x &= Ax + B(Ky + v)\,, \qquad y = Cx \\
\sigma x &= Ax + B(KCx + v)\,, \quad y = Cx \\
\sigma x &= (A + BKC)x + Bv\,, \quad y = Cx
\end{aligned}$$

- The design parameter in this case is the matrix K
- In SISO systems, K is a scalar (e.g. root locus method which is used to find closed loop poles)
- In MIMO systems, K is a matrix *(therefore, K can not be moved as the order of the multiplication matters)*

Static Output Feedback: vs Positive feedback

A common belief is that "Negative feedback is good, positive feedback is bad"

But if we look at the closed-loop system equations, what does 'negative feedback' actually mean?

$$\sigma x = (A + BKC)x + Bv \ , \quad y = Cx$$

If we write a minus sign in front of a gain K, does that automatically make it negative feedback? Well, what if K is negative, then get $-(-K)$, which is now positive feedback. **In state-space models, introducing a minus sign just complicates things unnecessarily.**

For transfer functions, negative feedback is well-defined due to the minus sign in the comparator. However, for state-space modelling, negative feedback is just a matter of convention and not a physical property.

Root Locus Method for SISO Systems

You will have learned in 'Control Systems' that for a single-input single-output (SISO) system, we can design controllers using the root locus method.

The objective is to find the gain K that places the closed-loop poles in desired locations.

The root locus is a powerful tool but is often misunderstood:

- By adjusting K, you can change some eigenvalues of the closed-loop system, but you cannot do very much.
- Since there is only one design parameter, K, modifying one pole forces the others to change.
- Therefore, only one eigenvalue can be directly controlled at a time.

Therefore, the root locus method is a very restricted design.

Adding Design Parameters: Static State Feedback

Instead of output feedback, we can use static state feedback.

Here, we modify system input in the following way

$$u = Kx + v$$

Therefore, the closed-loop system equations are

$$\sigma x = Ax + B(Kx + v) \, , \quad y = Cx$$
$$\sigma x = (A + BK)x + Bv \, , \quad y = Cx$$

where for SISO systems, K is a $n \times 1$ vector, and BK is a $n \times n$ matrix.

So, although it is not yet clear whether we can modify all of the eigenvalues of A *(we will see later on that this is possible in some situations)*, it is clear that this gives us more flexibility, but at a cost *(it requires full-state measurement)*.

Dynamic Output Feedback

We want to find a strategy that combines the simplicity of static output feedback with the flexibility (and power) of static state feedback.

To achieve this, we use dynamic output feedback, which uses a new dynamical system to process the output before feeding it back.

$$\sigma\xi = F\xi + Gy$$

$$u = K\xi + v$$

This new system is called the 'Controller' of your system, which is decoupled from your original system.

This setup is related to lead-lag compensation, where poles and zeros are introduced in the controller to achieve desired system characteristics, which some of you may have studied in the past.

Design Parameters: Dynamic Output Feedback

This new dynamical system (controller)

$$\sigma\xi = F\xi + Gy$$

$$u = K\xi + v$$

introduces new four design parameters: F, G, K, and the dimension of ξ.

In terms of the transfer function (which you studied in the 'Control Systems' module), the dimension of ξ is related to the number of 'poles' that you put into your controller (e.g. a PI controller will have a pole for $x = 0$ which means the $\dim \xi = 1$).

Note that $\dim \xi$ can go all the way up to n or even larger than n depending on your design objective.

In this module, we will focus on the case when $\dim \xi = n$, which is enough for the stabilization of the closed-loop system.

Closed-Loop System: Dynamic Output Feedback

The closed-loop system, combining plant and dynamic controller

$$\begin{aligned} \sigma x &= Ax + B(K\xi + v) \\ \sigma\xi &= F\xi + G(Cx) \\ y &= Cx \end{aligned}$$

where we have removed the matrix D to make the equations simpler.

In matrix form, we get

$$\begin{bmatrix} \sigma x \\ \sigma\xi \end{bmatrix} = \begin{bmatrix} A & BK \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v \,, \qquad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}$$

We will study static state feedback this week and dynamic output feedback next week.

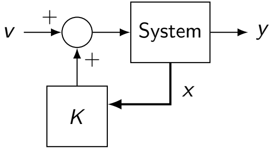
Note in this module, we will not cover dynamic state feedback, but it does exist and does have use cases.

Static State Feedback: Observability

We now want to find out what properties of the system are invariant with respect to feedback and what properties change.

Given the system

$$\sigma x = Ax + Bu$$
$$y = Cx$$
$$u = Kx + v$$



Rewriting these equations together, I get

$$\sigma x = Ax + BKx + Bv \,, \qquad y = Cx$$

Notice, straight away, that **observability** remains unchanged under static state feedback because it does not depend on the input.

Static State Feedback: Reachability

But is **Reachability** preserved under static state feedback?

Suppose that the original open loop system is reachable. This means the

$$\text{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n.$$

which is equivalent to the $\text{rank}(\left[\begin{array}{c|c} sI - A & B \end{array}\right]) = n\,, \quad \forall s \in \mathbb{C}$

Now we know that the rank of a matrix is not modified by the multiplication with another matrix of full rank. Therefore,

$$\text{rank}\left(\left[\begin{array}{c|c} sI - A & B \end{array}\right]\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\right) = n\,, \quad \forall s \in \mathbb{C}$$

which is equivalent to

$$\text{rank}(\left[\begin{array}{c|c} sI - \underbrace{(A + BK)}_{\substack{\tilde{A} \text{ of the} \\ \text{closed-loop system}}} & B \end{array}\right]) = n\,, \quad \forall s \in \mathbb{C} \quad \Longleftrightarrow \quad \text{The closed-loop system is reachable}$$

Static State Feedback: Reachability

So, we have now proved that reachability is not lost by feedback, but can reachability be gained by feedback?

Suppose that the original open loop system, $\sigma x = Ax + Bu$, is unreachable. This means the system can be decomposed into

$$\sigma \hat{x} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u \,, \qquad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

Now we apply feedback where $u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + v$ and we obtain

$$\sigma \hat{x} = \begin{bmatrix} \hat{A}_{11} + \hat{B}_1 K_1 & \hat{A}_{12} + \hat{B}_1 K_2 \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} v$$

Therefore, if the open loop system is unreachable $\iff$ the closed-loop system is unreachable **(where the unreachable modes are unchanged by feedback. Hence why they are often called the 'fixed' modes)**.

Role of Static State Feedback

So why is static state feedback useful?

- Feedback can modify part or completely the A matrix, shaping the system's dynamical behaviour.
- Recall the eigenvalues of A determine stability and transient properties.
- So modifying eigenvalues (or poles) enables faster response, reduced oscillations, or stabilization.

So how do we go about modifying the eigenvalues of the system?

Suppose we are given the following reachable system where

$$\begin{aligned} \sigma x &= Ax + Bu \\ u &= Kx + v \\ p &= 1 \text{ (i.e. a single input system)} \end{aligned}$$

Static State Feedback: Controllable Canonical Form

Since the system is reachable, we can write it in controllable canonical form

$$\sigma x = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} u$$

where

$$u = \begin{bmatrix} K_1 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + v$$

Therefore, applying feedback $u = Kx + v$ modifies the last row in the following way

$$\sigma x = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_0 + K_1 & -\alpha_1 + K_2 & \cdots & -\alpha_{n-1} + K_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} v$$

Static State Feedback: Characteristic Polynomial

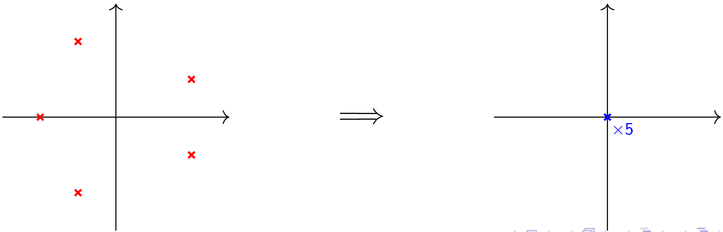
So, how can we now modify the characteristic polynomial of the system.

Suppose we make $K_1 = \alpha_0$, $\quad K_2 = \alpha_1$, $\quad K_3 = \alpha_2$, $\quad \cdots \quad K_n = \alpha_{n-1}$

Then, we get a new A matrix, which is

$$A_{\text{cl}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and have effectively moved all the eigenvalues to zero (the origin):



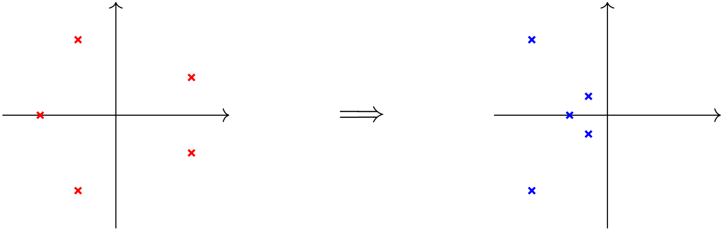
Static State Feedback: Pole (Eigenvalue) Placement

However, if with K, we can make all eigenvalues zero. Then, we can add an additional term to K and place the eigenvalue wherever we want.

So now we make $K_1 = \alpha_0 - \tilde{\alpha}_0$, $K_2 = \alpha_1 - \tilde{\alpha}_1$, $\cdots$ $K_n = \alpha_{n-1} - \tilde{\alpha}_{n-1}$.

This gives us a new A_{cl} matrix, which is $A_{\text{cl}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \tilde{\alpha}_0 & \tilde{\alpha}_1 & \cdots & \tilde{\alpha}_{n-1} \end{bmatrix}$

Now, we can move all the eigenvalues to wherever we want, for example:



Pole Placement and Ackermann's Formula

Therefore if we have now proved that in the case where $p = 1$

$$\text{Reachability} \quad \Longleftrightarrow \quad \begin{array}{c}\text{The eigenvalues of } A+BK \\ \text{can be arbitrarily assigned}\end{array}$$

There is actually a formula *(which is in the lecture notes)* called 'Ackermann's formula', and it provides a systematic way to compute K.

However, for low-order systems, i.e. if $n = 2$ or $n = 3$, it is more convenient to compute directly the characteristic polynomial of $A + BK$ and then compute K using the principle of identity of polynomials.

Note that K for each eigenvalue assignment is unique. However, K is not unique for multi-input systems.

Stabilizable Systems

A few things to note about stabilizable systems:

- Feedback shapes behaviour but does not alter reachability properties.
- If the system is not reachable, some eigenvalues remain fixed.
- A systems can be made stable (we say the system is **stabilizable**) even if not fully reachable where the fixed/unreachable modes are stable.

Next Steps

Next week, we will extend these concepts to dynamic and output feedback systems.

Coursework 2 will be released this Friday (21st March 2025)!