Advanced Control Systems

Lecture 7: Observability and Reconstructability

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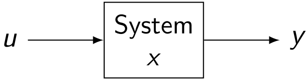
State-to-Output Properties

The idea is similar to input-to-state properties.

We consider a system with input u and output y.

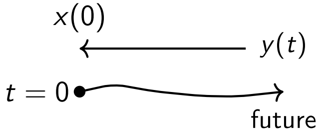


The goal is to quantify the information on state, $x(t)$, that can be obtained from measurements of the output signal, $y(t)$, over a given interval.

Two Perspectives: Observability (Future Measurements)

Much like reachability and controllability, we analyse the system from two perspectives

1. Using present and future output measurements to infer the current state.



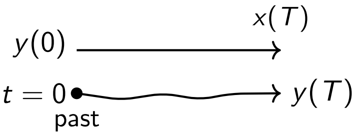
$y(t)$ is known for $t \leq 0$
$\implies$ find $x(0)$

- We use **observations** of $y(t)$ at the current time and in the future to help us estimate the initial state.
- This approach is common in online applications:
  - Real-time control, signal processing and communication applications
- This property is called **'obserability'** and relies on observations of future data to infer the initial state.

Two Perspectives: Reconstructability (Past Measurements)

Much like reachability and controllability, we analyse the system from two perspectives

2. Using past and present output measurements to infer the future state.



$y(t)$ is known for $t \in [0, T]$
$\implies$ find $x(T)$.

- We use $y(t)$ and the current time and in the past to help us **reconstruct** the state at $x(T)$
- This is useful in:
  - Weather forecasting and state estimation in dynamical systems
- This property is called **'Reconstructability'** and relies on historical data to infer the current state.

Observability vs Reconstructability

Similar to reachability and controllability in that **observability and reconstructability are the same if we reverse the arrow of time**.

- Observability and reconstructability are equivalent in continuous-time systems but can differ in discrete-time systems.
- In discrete-time systems, they differ due to eigenvalues at zero (as there may be several past for the same future).

In this module, we will focus all our efforts on the observability property, but it is good to know that the reconstructability property exists.

Reachability vs Observability

When discussing reachability and controllability, we have always taken a positive approach.

- I would like to go from A to B, and then I would like to find an input signal such that I can move the state of the system from A to B

However, in the case of the observability property (and reconstructed property), we have to identify a negative property.

We say that the system satisfies an observability condition if the system does not possess a negative property.

Observability in Linear Discrete-Time Systems

System equations are

$$x^+ = Ax + Bu \,, \qquad y = Cx$$

Given $y[k],\ k \geq 0$, and $u[k],\ k \geq 0$ then output for two states x_a, x_b is

$$\text{Pick } x_a[k] \implies y_a[k] = \underbrace{CA^k x_a[0]}_{\text{free response}} + \underbrace{\sum_{i=0}^{k-1} CA^{k-1-i} Bu[i] + Du[k]}_{\text{forced response}}$$

$$\text{Pick } x_b[k] \implies y_b[k] = \underbrace{CA^k x_b[0]}_{\text{free response}} + \underbrace{\sum_{i=0}^{k-1} CA^{k-1-i} Bu[i] + Du[k]}_{\text{forced response}}$$

Notice that x_a and x_b only appears in the free response of the state of the system. Therefore, if I measure $u[k]$ and $y[k]$, then the only important part I have to study is the free response of the output.

Indistinguishable States

So we now know that

- Observability depends only on the pair (A, C).
- We assume $u[k] = 0$ to focus on the free response of the system.

Therefore

$$x_a \implies y_a[k] = CA^k x_a,$$
$$x_b \implies y_b[k] = CA^k x_b.$$

Note that x_a and x_b are non-distinguishable for 0 steps in the future if $y_a[0] = y_b[0]$.

Likewise, x_a and x_b are non-distinguishable for 1 step in the future if $y_a[0] = y_b[0]$ and $y_a[1] = y_b[1]$.

and so on

Recursive Condition for Indistinguishability

Note that two states are indistinguishable in zero steps if
$y_a[0] = y_b[0] \iff Cx_a = Cx_b \implies C(x_a - x_b) = 0.$

In one step if $C(x_a - x_b) = 0$ and
$y_a[1] = y_b[1] \iff CAx_a = CAx_b \implies CA(x_a - x_b) = 0.$

In general, we can say that two states are indistinguishable in n steps if

$$\begin{aligned} C(x_a - x_b) &= 0, \\ &\vdots \\ CA^{n-1}(x_a - x_b) &= 0. \end{aligned}$$

We can now stop at $n-1$ steps because of the Cayley-Hamilton theorem, which tells us that if $CA^{n-1}(x_a - x_b) = 0 \implies CA^n(x_a - x_b) = 0$ because A^n can be written as a linear combination of all lower powers of A.

Kernel of the Observability Matrix

$$x_a \, , \, x_b \text{ indist. } 0 \iff C(x_a - x_b) = 0 \iff x_a - x_b \in \ker C$$

$$x_a \, , \, x_b \text{ indist. } 1 \iff CA(x_a - x_b) = 0 \iff x_a - x_b \in \ker \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$x_a \, , \, x_b \text{ indist. } 2 \iff CA^2(x_a - x_b) = 0 \iff x_a - x_b \in \ker \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$\vdots$$

$$x_a \, , \, x_b \text{ indist. n} \iff CA^{n-1}(x_a - x_b) = 0 \iff x_a - x_b \in \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A kernel is just the set of all vectors, and when you multiply the vector with the matrix, you get zero. *It is just a name!*

Observability Matrix

This gives us the **observability matrix** of the system

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The observability matrix of the system is used to express the fact that two states may be distinguishable or non-distinguishable.

If the difference $x_a - x_b$ belongs to $\ker \mathcal{O}$, which means the product of $\mathcal{O}$ with $x_a - x_b$ is equal to zero, then this **pair is not distinguishable**.

If the difference $x_a - x_b$ does not belong to $\ker \mathcal{O}$, which means the product of $\mathcal{O}$ with $x_a - x_b$ is different from zero, then the **pair is distinguishable**.

Properties of the Observability Matrix

So for the pair x_a , x_b to be distinguishable, there needs to be at least one sample from each of their outputs which is different.

So, we can use this observability matrix of the system to express a property of the overall system.

$$\mathcal{O} = \left.\overbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}^{n \text{ columns}}\right\} q \times n \text{ rows}$$

Note that if $q = 1$ (i.e. a single output system), then the observability matrix is square, $n \times n$.

It is interesting to observe how similar some of the tools we are using here are to the ones we used for reachability and controllability.

Full Rank Condition for Observability

So we know that x_a is indistinguishable (regardless of the number of steps) from x_b if $x_a - x_b \in \ker \mathcal{O}$.

This means that

- A system is **observable** if and only if $\mathcal{O}$ has full rank n
- When $q = 1$ then $\det(\mathcal{O}) \neq 0$
- $\iff$ all pairs x_a , x_b are non-indistinguishable (double negative)

We could simplify this further by saying that a state $\bar{x}$ is indistinguishable (regardless of the number of steps) from 0 if $\bar{x} \in \ker \mathcal{O}$.

- All states $\bar{x}$ can be distinguishable from 0 (regardless of the number of steps)

Observability in Practice

- The output represents what the designer chooses to measure.
- For a mechanical system, we could measure positions, velocities, accelerations, etc.
- However, the more outputs we measure, the higher the cost.
- For example, measuring only velocities may lose positional information, but a clever combination can restore observability.
- In a multiple-output system, removing the wrong output may lead to loss of observability.

Example 1: A Non-Observable System

Given:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

The observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\det(\mathcal{O}) = 0$, the system is non-observable.

Example 1: Geometric Interpretation

The kernel consists of vectors $v = (v_1, v_2, v_3)^T$ such that:

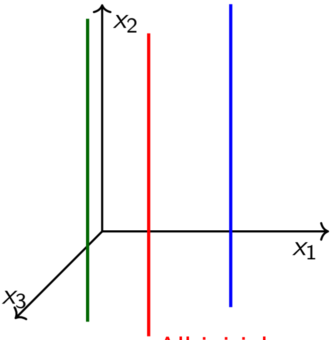
$$\mathcal{O}v = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_3 \\ v_3 \\ v_3 \end{bmatrix} = 0.$$

This implies $v_3 = 0$ and $v_1 = 0$, leaving v_2 free:

$$\ker(\mathcal{O}) = \operatorname{im}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

Sometimes call the 'span' i.e. $\ker(\mathcal{O}) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

Example 1: A Non-Observable System



All initial states alone this line generate the same output

That is to say, the states are indistinguishable along any line parallel to the x_2-axis.

Observable System

Suppose $q = 1$ and rank $\mathcal{O} = n$ (full rank) $\iff$ $\det(\mathcal{O}) \neq 0$

Now, suppose I measure

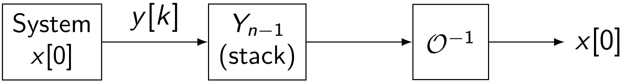
$$
\begin{aligned}
y[0] &= Cx[0] \\
y[1] &= CAx[0] \\
&\vdots \\
y[n-1] &= CA^{n-1}x[0]
\end{aligned}
\qquad
Y_{n-1} = \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} = \mathcal{O}x[0]
$$

Therefore the state $x[0]$ is given by

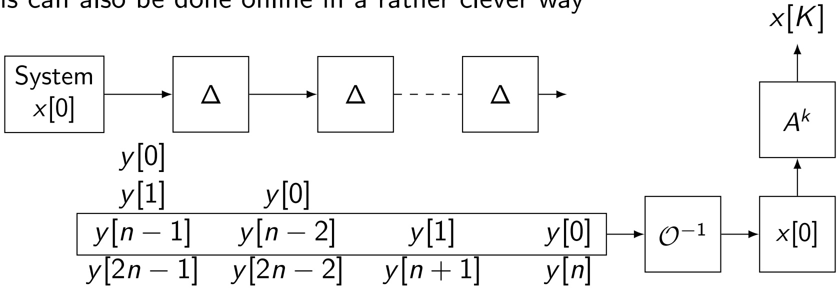
$$x[0] = \mathcal{O}^{-1} Y_{n-1}$$

Observability Block Diagram

We can now represent this equation as a block diagram in the following way



This can also be done online in a rather clever way



No feedback here, so this method is very sensitive noise.

Next Steps: Observability in Continuous-Time Systems

- We will now explore observability for continuous-time systems.
- The link between observability and reachability remains strong.
- The techniques developed for discrete-time systems extend naturally to the continuous case.

Continuous-Time Systems

Given a continuous-time system

$$\dot{x} = Ax + Bu \,, \quad y = Cx$$

Note again that D is not important as it does not reveal anything about the internal properties of the system.

If we measure $u(t)$ for all $t > 0$ and $y(t)$ for all $t > 0$ then we can effectively measure the free response of the output

$$Ce^{At}x(0)$$

Remember we can always assume that input signal is identically equal to zero for observability analysis.

Observability in Continuous-Time Systems

We first need to study a negative property. We must verify when two initial start states cannot be distinguished from output measurements.

Once we have established this property, then we will call a system 'observable' if it does not possess this indistinguishable property

Formally:

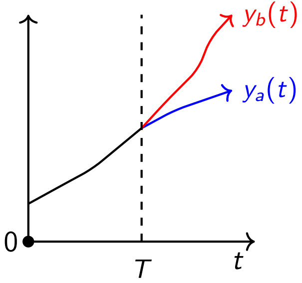
- Two states x_a and x_b are indistinguishable in some interval $[0, T]$, $T > 0$ if

$$y_a(t) = Ce^{At}x_a = Ce^{At}x_b = y_b(t), \quad \forall t \in [0, T]$$

- That is to say that their output trajectories coincide over $[0, T]$.

Indistinguishability of Continuous-Time Systems

$$y_a(t) = Ce^{At}x_a = Ce^{At}x_b = y_b(t), \quad \text{for all } t \in [0, T]$$



Now, in principle, $y_a(t)$ and $y_b(t)$ could have a difference in the future.

However, we will see that if you are unable to distinguish between these two states in the interval $[0, T]$, then you will never be able to distinguish between these two states.

Indistinguishability Condition

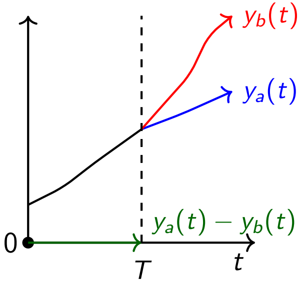
Rewriting we can say, two states x_a and x_b are indistinguishable in some interval $[0, T]$, $T > 0$ if

$$y_a(t) - y_b(t) = 0 \ , \quad \forall t \in [0, T]$$

(zero function)

$$Ce^{At}x_a - Ce^{At}x_b = 0 \ , \quad \forall t \in [0, T]$$

$$Ce^{At}(x_a - x_b) = 0 \ , \quad \forall t \in [0, T]$$



First, notice that if a function is zero in some interval $[0, T]$, then all of its time derivative will also be zero.

Second, recall that e^{At} is an analytic function (That is to say, e^{At} is equal to its Taylor series expansion.)

Indistinguishability Condition: Observability Matrix

Using these two facts we can now say two states x_a and x_b are indistinguishable in some interval $[0, T]$, $T > 0$ if

$$\frac{d^k}{dt^k}\left[\ Ce^{At}(x_a - x_b)\right]\Big|_{t=0} = 0 \quad \text{for all } k \geq 0 \text{ where } k \text{ is an integer}$$

Therefore, if we write out this condition, we get

$$\begin{aligned} C(x_a - x_b) &= 0\ , \quad \text{since } e^0 = I \\ CA(x_a - x_b) &= 0\ , \quad \text{since } \frac{d}{dt}e^{At} = Ae^{At} \\ &\vdots \\ CA^{n-1}(x_a - x_b) &= 0 \quad \text{which is } \frac{d^{n-1}}{dt^{n-1}}\left[\ Ce^{At}(x_a - x_b)\right]\Big|_{t=0} \end{aligned}$$

We can again stop at $n - 1$ because of the Cayley-Hamilton theorem.

Observability Matrix

We can now summarise by saying two states x_a and x_b are indistinguishable in some interval $[0, T]$, for all $T > 0$ if

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} (x_a - x_b) = 0$$

This gives us the same 'observability matrix', $\mathcal{O}$, as for discrete-time systems.

Notice that since e^{At} is analytic, we have only been looking at properties for $t = 0$. This means that our interval can be any positive length.

Observability

We can now set $x_a = \bar{x}$ and $x_b = 0$.

Therefore, we can say a continuous-time system

$$\dot{x} = Ax + Bu \,, \quad y = Cx$$

is observable if all states can be distinguished from the zero state.

Which means

$$\text{Observability} \iff \text{rank}\, \mathcal{O} = \text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Energy: Observability Gramian

We are not going to cover the Observability Gramian in the module, but it is interesting to take a quick look

Observability Energy: Similar to reachability, we can define an energy measure for observability.

- Reachability Gramian: Measures energy needed to reach a state.
- Observability Gramian: Measures energy stored in the initial state.
- Defined as:

$$V_t = \int_0^t \underbrace{e^{A'\tau} C'}\underbrace{C e^{A\tau}} d\tau$$

We can now add to the observability conditions

$$\text{Observability} \iff \text{rank}\, \mathcal{O} = n \iff V_t > 0\,,\ \forall\, t > 0 \implies \det(V_t) \neq 0$$

i.e if V_T is strictly positive definite, the system is observable.

Interpretation of V_T

How can we understand the meaning of the Observability Gramian, V_t.

To understand the meaning we

$$
\begin{aligned}
x(0)'V_T x(0) &= \int_0^t x(0)'e^{A'\tau}C'Ce^{A\tau}x(0)d\tau \\
&= \int_0^t y'(t)y(t)d\tau \\
&= \int_0^t \|y(t)\|^2 dt \quad \text{(Energy of the output of the system)}
\end{aligned}
$$

- This represents the energy of the output signal
- This implies that the only state yielding zero energy is $x(0) = 0$ (provided the system is observable)

Block Diagram: Extracting the Initial State

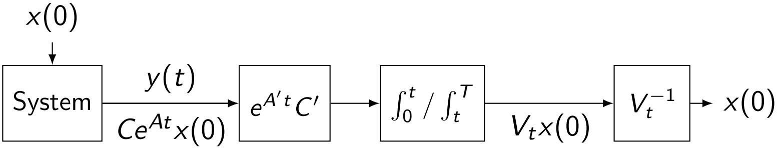
Given a continuous-time system

$$\dot{x} = Ax\;, \quad y = Cx$$

Recall that

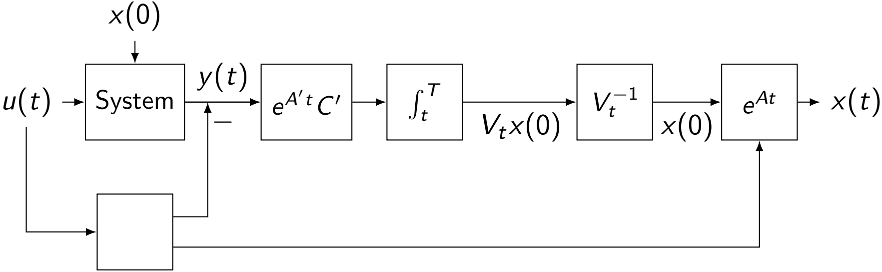
$$V_t = \int_0^t \underbrace{e^{A'\tau}C'}_{}\,\underbrace{Ce^{A\tau}}_{}\,d\tau$$

Then, we can construct the following block diagram



Block Diagram: Extracting the Initial State with Input

If we have an input signal, we simply need to cancel it at the output and add the forced response at the end.



Challenges:

- Noise sensitivity.
- Precise cancellations are needed.

We will see an alternative approach to constructing an estimate of the state in a later lecture.

Structural Properties

We have now seen two *(four for discrete-time systems)* structural properties

- Reachability $\rightarrow [B, AB, A^2B, \ldots, A^{n-1}B]$ (wide matrix)
- Observability $\rightarrow \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ (tall matrix)

Notice that both matrices are constructed by multiplying powers of A iteratively with B or C.

Now recall that the transposition of a matrix multiplication is

$$(XY)' = Y'X'$$

Matrix Transposition

Therefore the transpose of the observability matrix, $\mathcal{O}$, gives us

$$\mathcal{O}' = \begin{bmatrix} C' & A'C' & (A')^2C' & \cdots \end{bmatrix}$$

which is just the reachability matrix for another system $\sigma\xi = A'\xi + C'v$.

Likewise, the transpose of the reachability matrix, R, gives us

$$R' = \begin{bmatrix} B' \\ B'A' \\ B(A')^2 \\ \vdots \end{bmatrix}$$

which is just the observability matrix for another system $\sigma\xi = A'\xi$ and $\eta = B'\xi$.

We have just transposed our four matrices A, B, C and D and swapped the roles of B and C.

System Representation and Duality

Given the original system:

$$\sigma x = Ax + Bu, \quad y = Cx + Du \quad \text{(often called the primal system)}$$

We define

$$\sigma \xi = A'\xi + C'v \quad \eta = B'\xi + D'u \qquad \text{(often called the dual system)}$$

If we write the systems in complete matrix form we get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{primal system}$$

$$\begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix} \quad \text{dual system}$$

Note that the dual of the dual system returns to the original system.

PBH Test for Observability

Give the system

$$\sigma x = Ax, \quad y = Cx \underset{\text{Dual}}{\Longleftrightarrow} \sigma \xi = A'\xi + C'v$$

I now know that the dual system is reachable if

$$\text{rank}\left[\begin{array}{c|c} sI - A' & C' \end{array}\right] = n\,, \quad \text{for all } s \notin P_\lambda(A')$$

But recall that whatever is reachable for the dual system is observable for the original system.

Therefore, the PBH test for Observability is given by

$$\text{rank}\begin{bmatrix} sI - A \\ C \end{bmatrix} = n\,, \qquad \text{for all } s \notin P_\lambda(A)\,, \quad \text{recall } P_\lambda(A') = P_\lambda(A)$$

Canonical Form for Observable Systems

An observable system is algebraically equivalent to:

$$\sigma \hat{x} = A_o \hat{x} + B_o u, \quad y = C_o \hat{x} + D_o u,$$

where:

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}, \quad C_o = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are the coefficients of the characteristic polynomial.

This canonical form highlights the structure of observable systems.

System Decomposition

We now want to analyse the decomposition of a system

$$\sigma x = Ax \,, \quad y = Cx$$

that is not observable.

This implies that the dual system

$$\sigma \xi = A' \xi + C' v$$

is non-reachable.

There in some coordinates I can split the state ξ into two parts

$$\begin{bmatrix} \sigma \xi_1 \\ \sigma \xi_2 \end{bmatrix} = \begin{bmatrix} A'_{11} & A'_{21} \\ 0 & A'_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} C'_1 \\ 0 \end{bmatrix} v$$

Then, taking the dual

$$\begin{bmatrix} \sigma z_1 \\ \sigma z_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

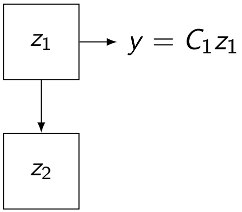
Observable and Unobservable Modes

Now we can say that the

- The unobservable modes of the system are the eigenvalues of A_{22} for which the observability matrix loses rank.
- The observable modes of the system are the eigenvalues of A_{11} for which the observability matrix loses rank.

$$\sigma z_1 = A_{11} z_1$$
$$\sigma z_2 = A_{21} z_1 + A_{22} z_2$$

$$y = C_1 z_1$$



Observable Subsystem

Non-Observable Subsystem

Kalman Decomposition

Kalman Decomposition is not covered in this module but is in the notes.

- Kalman Decomposition shows that the system can be partitioned into four components:
  - Reachable and Observable
  - Reachable and Unobservable
  - Unreachable and Observable
  - Unreachable and Unobservable

The problem with transfer function representations is that they only model observable and reachable components.

The state-space approach provides deeper insight into system properties.

Implications for Control Design

In the coming lectures, we are going to look at 'Control Design'

- Control design exploits reachability and observability.
- Two feedback design approaches:
  - State feedback (requires full state information)
  - Output feedback (requires state estimation)
- Introduction to observers and filters for state estimation.

Next week, we will also review the answers for the QMPlus (coursework 1) Quiz in the second part of the lecture.