Advanced Control Systems

Lecture 6: Reachability and Controllability (Canonical Form and PBH Reachability Test)

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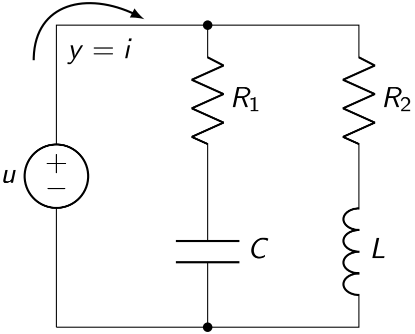
Electrical Network Example

Let's see how reachability plays out in practice by studying the reachability properties of an electrical network:



The input u is the voltage across the voltage source, and the output y is the current delivered by the voltage source.

Electrical Network Example: System Equations

We define the state variables

- x_1: voltage across the capacitor C
- x_2: current through the inductor L

Kirchhoff's laws yield

$$u = x_1 + R_1 C \dot{x}_1 \, , \qquad u = R_2 x_2 + L \dot{x}_2 \, , \qquad y = i = x_2 + \frac{u - x_1}{R_1}$$

The system is, therefore, described by the equations

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{R_1 C} x_1 + \frac{1}{R_1 C} u \\ \dot{x}_2 &= -\frac{R_2}{L} x_2 + \frac{1}{L} u \end{aligned}$$

The output equation is

$$y = -\frac{1}{R_1} x_1 + x_2 + \frac{1}{R_1} u$$

Electrical Network Example: State-Space Equations

As a result, the state-space equations are

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix} u$$

and

$$y = Cx + Du$$

$$y = \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{R_1} u.$$

The system matrices are:

$$A = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix}, \quad C = \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix}, \quad D = \mathbf{0}$$

Electrical Network Example: Reachability Matrix

$$A = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix}, \quad C = \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix}, \quad D = \mathbf{0}$$

The reachability matrix is given by:

$$R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 C} & -\frac{1}{R_1^2 C^2} \\ \frac{1}{L} & -\frac{R_2}{L^2} \end{bmatrix}.$$

The determinant of R is:

$$\begin{aligned} \det(R) &= \left(\frac{1}{R_1 C}\right)\left(-\frac{R_2}{L^2}\right) - \left(\frac{1}{L}\right)\left(-\frac{1}{R_1^2 C^2}\right) = \frac{1}{L R_1^2 C^2} - \frac{R_2}{R_1 C L^2} \\ &= \frac{1}{R_1 C L}\left(\frac{1}{R_1 C} - \frac{R_2}{L}\right) \end{aligned}$$

Electrical Network Example: Reachability Condition

The system is reachable (and controllable) if

$$\det(R) = \frac{1}{R_1 CL}\left(\frac{1}{R_1 C} - \frac{R_2}{L}\right) \neq 0$$

Therefore, the system is reachable (and controllable) if

$$\frac{1}{R_1 C} \neq \frac{R_2}{L}$$

and the system is not reachable (and not controllable) if

$$\frac{1}{R_1 C} = \frac{R_2}{L}$$

Electrical Network Example: Physical Interpretation

We have shown that the system is not reachable (and not controllable) if

$$\frac{1}{R_1 C} = \frac{R_2}{L}$$

But how does this relate to the physical system?

Notice that the time constants of the system are:

$$\tau_1 = R_1 C \; , \quad \text{(time constant across } \frac{1}{R_1 C} \text{ component of the circuit)}$$
$$\tau_2 = \frac{L}{R_2} \; , \quad \text{(time constant across } \frac{R_2}{L} \text{ component of the circuit)}$$

If $\tau_1 = \tau_2$, the system is unreachable because we are not able to independently control the voltage across the capacitor and the current through the inductor.

Electrical Network Example: System Equations

If we go back to the system equations, we can see this

$$\dot{x}_1 = -\frac{1}{R_1 C} x_1 + \frac{1}{R_1 C} u$$
$$\dot{x}_2 = -\frac{R_2}{L} x_2 + \frac{1}{L} u$$

If $\frac{1}{R_1 C} = \frac{R_2}{L}$, then the two equations have the same coefficient for x_1 and x_2 so system can be rewritten as:

$$\dot{x}_1 = -\lambda x_1 + b_1 u$$
$$\dot{x}_2 = -\lambda x_2 + b_2 u$$

If we multiply both equations by b_2 and b_1 respectively, we get

$$b_2 \dot{x}_1 = -b_2 \lambda x_1 + b_2 b_1 u$$
$$b_1 \dot{x}_2 = -b_1 \lambda x_2 + b_1 b_2 u$$

If we now add these two equations

$$b_2 \dot{x}_1 - b_1 \dot{x}_2 = -b_2 \lambda x_1 + b_1 \lambda x_2$$

Electrical Network Example: System Equations

Now we can write this as

$$\overbrace{b_2x_1 - b_1x_2}^{\cdot} = -\lambda(b_2x_1 - b_1x_2)$$

We can now substitute in z to obtain

$$\dot{z} = -\lambda z, \quad \text{where } z = b_2x_1 - b_1x_2$$

Therefore, z (as we have seen in a previous lecture) is nothing more than

$$z = z(0)e^{-\lambda t}$$

That is to say that

$$b_2x_1 - b_1x_2 = z(0)e^{-\lambda t}$$

As $t \to \infty$, $z(0)e^{-\lambda t} \to 0$, therefore

$$x_1 = \frac{b_1}{b_2}x_2 \quad \text{(the } v \text{ across } C \text{ is not independent of the } i \text{ through } L\text{)}$$

Electrical Network Example: Implications of reachability

- The system is not reachable/controllable because the voltage across the capacitor and the current through the inductor cannot be independently controlled
  - As $t \to \infty$, $x_1(t) = \frac{B_1}{B_2} x_2(t)$
- Notice that small variations (e.g. temperature drift in resistances) can restore reachability/controllability
  - Energy considerations: if the system is nearly unreachable/uncontrollable, control requires high energy
  - Small perturbations lead to $\dot{z} = -\lambda z + \epsilon u$ (notice that a large input signal would be needed for small ϵ)
- Controllability is often termed a 'generic property' as almost all randomly chosen systems are controllable

Canonical Controllable Form: System Representation

Given a system (continuous or discrete time):

$$\sigma x = Ax + Bu$$

- Single input assumption: $p = 1 \Rightarrow B$ is a column vector
- The system is assumed to be reachable

Reachability matrix:

$$R = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

Since the system is reachable, R is square and invertible such that

$$\det(R) \neq 0$$

Canonical Controllable Form: Finding Vector L

We seek to find a L such that:

$$LR = L\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

This implies:

$$LB = 0, \quad LAB = 0, \quad \cdots \quad LA^{n-2}B = 0, \quad LA^{n-1}B = 1$$

Therefore, we can write L as

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} R^{-1}$$

Canonical Controllable Form: New Coordinate System

We now define a new coordinate system that exploits the properties of L

$$z_1 = Lx, \quad z_2 = LAx, \quad \cdots \quad z_n = LA^{n-1}x$$

In matrix form:

$$z = Tx = \overbrace{\begin{bmatrix} L \\ LA \\ LA^2 \\ \vdots \\ LA^{n-1} \end{bmatrix}}^{\text{square matrix}} x$$

Now, we need to prove that T is invertible. Well, we know that

$$TR = \begin{bmatrix} L \\ \vdots \\ LA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

and R are both invertible. Therefore, T is invertible, since these are two square matices $\det(AB) = (\det A)(\det B)$

Canonical Controllable Form: New Coordinate System

Recall:

$$LB = 0, \quad LAB = 0, \quad \cdots \quad LA^{n-2}B = 0, \quad LA^{n-1}B = 1$$

$$TR = \begin{bmatrix} L \\ LA \\ LA^2 \\ \vdots \\ LA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & LA^nB \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & \cdots & \cdots \\ 1 & LA^nB & \cdots & \cdots & \cdots \end{bmatrix}$$

The determinant is $\det(TR) = -1^{n-1} = \pm 1$

New Coordinates

Now we could compute the transformed matrix $\hat{A} = TAT^{-1}$

- The issue is we need to know T^{-1}

Instead, we will take a different approach.

Recall that our new coordinates are defined as:

$$
\begin{aligned}
z_1 &= Lx, \\
z_2 &= LAx, \\
z_3 &= LA^2x, \\
&\vdots \\
z_{n-1} &= LA^{n-2}x, \\
z_n &= LA^{n-1}x.
\end{aligned}
$$

These are defined using the transformation matrix T.

Time Derivative

Now let's consider $\dot{z}_1 = L\dot{x}$:

$$\dot{z}_1 = L\dot{x} = L(Ax + Bu) = \underbrace{LAx}_{z_2} + \underbrace{LB}_{0} u = z_2$$

Now let's consider $\dot{z}_2 = LA\dot{x}$:

$$\dot{z}_2 = LA\dot{x} = LA(Ax + Bu) = \underbrace{LA^2x}_{z_3} + \underbrace{LAB}_{0} u = z_3$$

Applying the same logic:

$$\dot{z}_{n-1} = LA^{n-2}\dot{x} = LA^{n-2}(Ax + Bu) = \underbrace{LA^{n-1}x}_{z_n} + \underbrace{LA^{n-2}B}_{0} u = z_n$$

Until

$$\dot{z}_n = LA^{n-1}\dot{x} = LA^{n-1}(Ax + Bu) = LA^n x + \underbrace{LA^{n-1}B}_{=1} u = LA^n x + u$$

Applying Cayley-Hamilton

However, $\dot{z}_n$ can be simplified using the Cayley-Hamilton theorem.

Cayley-Hamilton theorem states that

$$A^n = -\alpha_{n-1}A^{n-1} - \cdots - \alpha_1 A - \alpha_0 I.$$

Therefore substituting into $\dot{z}_n$ gives

$$\dot{z}_n = L[-\alpha_{n-1}A^{n-1} - \cdots - \alpha_1 A - \alpha_0 I]x + u$$
$$\dot{z}_n = [-\alpha_{n-1}\underbrace{LA^{n-1}x}_{z_n} - \cdots - \alpha_1 \underbrace{LAx}_{z_2} - \alpha_0 \underbrace{Lx}_{z_1}] + u$$

Therefore

$$\dot{z}_n = -\alpha_0 z_1 - \alpha_1 z_2 - \cdots - \alpha_{n-1} z_n + u.$$

System Description

This system is described by:

$$
\begin{aligned}
\sigma z_1 &= z_2 \\
\sigma z_2 &= z_3 \\
&\vdots \\
\sigma z_{n-1} &= z_n \\
\sigma z_n &= -\alpha_0 z_1 - \alpha_1 z_2 - \cdots - \alpha_{n-1} z_n + u
\end{aligned}
$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are the coefficients of the characteristic polynomial.

Matrix Representation

The system can be rewritten in matrix form as:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Observations:

- The system is still reachable/controllable after a change of coordinates.
- Reachability/controllability is not altered by a coordinate transformation.
- This pair (A, B) is known as the controllability canonical form.
- The characteristic polynomial coefficients define the system dynamics.

Efficient Computation of Canonical Form

Rewriting the system in canonical form is actually now very straightforward.

Steps to derive the canonical form:

- Compute the reachability matrix.
- Check that it is full rank.
- Directly construct the canonical form without matrix inversion.
- Copy the characteristic polynomial coefficients directly into the last row.

Simple!

Feedback Control

This canonical form is useful because

$$\sigma z_n = -\alpha_0 z_1 - \alpha_1 z_2 - \cdots - \alpha_{n-1} z_n + u$$

and we can use feedback to set u:

$$u = \alpha_0 z_1 + \alpha_1 z_2 + \cdots + \alpha_{n-1} z_n + v$$

Which results in:

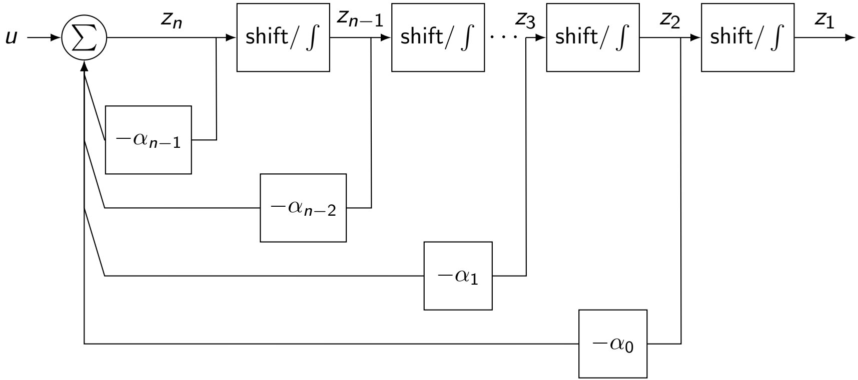
$$\dot{z}_n = v$$

We will see in later lectures why this is so useful, but it is because

- This feedback can cancel the characteristic polynomial terms.
- Therefore, the system can be modified dynamically using control.
- Enables pole placement.

Block Diagram Representation

The canonical form structure can also be represented by a block diagram:



- z_n propagates through the system through a series of integrators (or shift registers in discrete time).
- The characteristic polynomial coefficients define system behaviour.

Canonical Form for Non-Reachable Systems

We would now like to discuss a canonical form for non-reachable systems.

Suppose we are given a system

$$\sigma x = Ax + Bu$$

where the rank of the reachability matrix R is given by

$$\text{rank}(R) = \rho, \quad \rho < n$$

This implies that the system is not reachable.

Our aim now is to try and separate the reachable and non-reachable components of the system

Transformation of the System

To separate the reachable and non-reachable components, we define new coordinates $\hat{x}$

$$x = L\hat{x}$$

where transformation matrix L should incorporate information from the reachability matrix R.

Since R has ρ linearly independent columns, we construct L as:

$$L = \begin{bmatrix} L_1 & L_2 \end{bmatrix},$$

where:

- L_1 consists of ρ linearly independent columns from R (spanning the $\text{im}(R)$).
- L_2 consists of $n - \rho$ additional columns chosen to make L invertible.
  - A convenient choice for L_2 is columns with mostly zeros and a single one in each row

New System Representation

Recall that applying the coordinate transformation gives the system equations

$$\dot{\hat{x}} = \underbrace{L^{-1}AL}_{\hat{A}}\hat{x} + \underbrace{L^{-1}}_{\hat{B}}Bu$$

there the transformed system matrices are

$$\hat{A} = L^{-1}AL, \quad \hat{B} = L^{-1}B$$

To simplify the computation, we rewrite the equations to remove the inverse of L

$$L\hat{A} = AL, \quad L\hat{B} = B$$

Block Structure of $\hat{B}$

Since L_{11} spans the image of R, and B lies in the $\text{Im}(R)$, we partition L and analyse its effect on $\hat{B}$:

$$L\hat{B} = B \implies \begin{bmatrix} \underbrace{\begin{matrix} L_{11} \\ L_{21} \end{matrix}}_{\text{Im}(R)} & \begin{matrix} L_{12} \\ L_{22} \end{matrix} \end{bmatrix} \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where L_{11} is a square matrix with dimensions $\rho \times \rho$ and L_{22} is also a square matrix with dimensions $(n - \rho) \times (n - \rho)$

We know that $B \subset \text{Im}(R)$, therefore, $B_2 = 0$

$$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$$

Block Structure of $\hat{A}$

A similar analysis applied to $\hat{A}$ reveals a block structure

$$L\hat{A} = AL \implies \begin{bmatrix} \underbrace{\begin{matrix} L_{11} \\ L_{21} \end{matrix}}_{\text{Im}(R)} & \begin{matrix} L_{12} \\ L_{22} \end{matrix} \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \underbrace{\begin{matrix} L_{11} \\ L_{21} \end{matrix}}_{\text{Im}(R)} & \begin{matrix} L_{12} \\ L_{22} \end{matrix} \end{bmatrix}$$

First we see that if we multiply A_{11} by L_{11} we get $A\text{Im}(R) = \text{Im}(R)$ therefore

$$\begin{bmatrix} \underbrace{\begin{matrix} L_{11} \\ L_{21} \end{matrix}}_{\text{Im}(R)} & \begin{matrix} L_{12} \\ L_{22} \end{matrix} \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \underbrace{\begin{matrix} L_{11} \\ L_{21} \end{matrix}}_{\text{Im}(R)} & \begin{matrix} L_{12} \\ L_{22} \end{matrix} \end{bmatrix} = \left[\begin{array}{c|c} \text{Im}(R) & \quad \end{array} \right]$$

Now see that L_{12} multiply $\hat{A}_{21}$ will not be in the $\text{Im}(R)$.

So we need $\hat{A}_{21} = 0$.

System Representation using $\hat{A}$ and $\hat{B}$

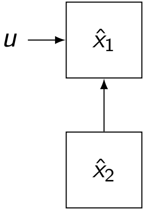
So, the new system can be written in the following form

$$\sigma \hat{x} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u$$

So, the equations of the system are

$$\sigma \hat{x}_1 = \hat{A}_{11} \hat{x}_1 + \hat{A}_{12} \hat{x}_2 + \hat{B}_1 u$$

$$\sigma \hat{x}_2 = \hat{A}_{22} \hat{x}_2$$



Reachable Subsystem (ρ states)

Non-Reachable Subsystem ($n - \rho$ states)

See that there is no connection between u and $\hat{x}_2$, so if

$$\hat{x}_2(0) = 0 \implies \hat{x}_2(t) = 0, \quad t \geq 0$$

Checking Reachability of the Reachable Subsystem

To verify that the reachable subsystem is indeed reachable, we must analyse the reachability matrix, $\hat{R}$ when

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$$

where $\hat{R} = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} & \cdots \end{bmatrix}$.

Therefore $\hat{R}$ is

$$\hat{R} = \left[\begin{array}{c|c|c|c} \hat{B}_1 & \hat{A}_{11}\hat{B}_1 & \hat{A}_{11}^2\hat{B}_1 & \cdots \\ 0 & 0 & 0 & \end{array} \right]$$

We know that $\text{rank}(R) = \rho \iff \text{rank}(\hat{R}) = \rho$ because $R = L\hat{R}$.

Checking Reachability of the Reachable Subsystem

Therefore rank$\left(\begin{bmatrix}\hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B}_1 & \cdots\end{bmatrix}\right) = \rho$ which means the system

$$\sigma\hat{x}_1 = \hat{A}_{11}\hat{x} + \hat{B}_1 u \quad \text{is reachable.}$$

Note that the $\hat{A}_{12}\hat{x}_2$ term will always be zero if you start at the origin.

Now, if we look at the eigenvalues, we know that

$$\lambda(A) = \lambda(\hat{A})$$

that is to say the eigenvalues of A are the same as the eigenvalues of $\hat{A}$. However, because the $\hat{A}$ is upper triangular, we can also say

$$\lambda(\hat{A}) = \lambda(\hat{A}_{11}) \cup \lambda(\hat{A}_{22})$$

- $\lambda(\hat{A}_{11})$ are often called the reachable modes of the system
- $\lambda(\hat{A}_{22})$ are often called the unreachable (or fixed) modes of the system

Fixed Modes and Control Implications

- Eigenvalues correspond to fixed modes that cannot be altered by feedback. Essentially these modes remain unchanged regardless of any control strategies
- A reachable mode is something that you may not see in the input-output behaviour of the system

- Interestingly, in terms of transfer functions, you will typically have an unreachable mode when you get a pole-zero cancellation
  - The evolution of these modes still needs to be controlled in some way, even if we don't see it externally
  - Hence, you would have been told in the 'control systems' module that pole-zero cancellations can only occur in the left half of the complex plane for continuous-time systems
  - If you cancel a pole-zero pair in the unstable region, you may get an unreachable mode that is unstable

PBH Test for Reachability

Characterizing the reachable mode is difficult because it requires constructing a matrix L and do a coordinate transformation.

However, there is a much simpler alternative to check reachability and it is called the **PBH test**.

The PBH test directly determines the presence of non-reachable modes without the need for matrix transformations.

Reachability Pencil

We are given the usual system:

$$\sigma x = Ax + Bu$$

Then, we define something called the *reachability pencil*.

The *reachability pencil* is a new matrix that is given by

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right]$$

which is a polynomial matrix with n rows and $n + p$ columns.

The PBH test states that we can check reachability directly by analysing the properties of the *reachability pencil*

$$\text{System is Reachable} \iff \text{rank}(\left[\begin{array}{c|c} sI - A & B \end{array}\right]) = n, \quad \forall s \in \mathbb{C}.$$

Reachability Condition

We now have a test for reachability

$$\text{System is Reachable} \iff \text{rank}(\left[\begin{array}{c|c} sI - A & B \end{array}\right]) = n, \quad \forall s \in \mathbb{C}.$$

However, checking for all $s \in \mathbb{C}$ is impractical since there are infinitely many values.

However, a key observation simplifies this:

- If s is not an eigenvalue of A, then $sI - A$ is full rank.
- **We only need to check for eigenvalues of A**

$$\text{rank}(\left[\begin{array}{c|c} sI - A & B \end{array}\right]) = n, \quad \forall s \in \lambda(A).$$

If there exists an $\bar{s}$ such that:

$$\text{rank}(\left[\begin{array}{c|c} \bar{s}I - A & B \end{array}\right]) < n$$

the system is not reachable, and $\bar{s}$ is an unreachable mode of the system.

Reachability Pencil Condition

If a system $\sigma x = Ax + Bu$ is not reachable, we would like to prove that

$$\text{rank}(\left[\begin{array}{c|c} sI - A & B \end{array}\right]) < n, \quad \text{for some } s$$

We know that if the system is not reachable if $\text{rank}(R) = \rho < n$.

and that can decompose it into reachable and unreachable components

$$\sigma \hat{x} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u$$

Then, the reachability pencil of the decomposed system is

$$\left[\begin{array}{cc|c} sI - \hat{A}_{11} & -\hat{A}_{12} & \hat{B}_1 \\ 0 & sI - \hat{A}_{22} & 0 \end{array}\right]$$

If $\bar{s}$ is an eigenvalue of A_{22}, then the reachability pencil loses rank (the bottom row is all zeros), confirming non-reachability and that $\text{rank}(R) < n$.

Advantages of the PBH Test

So why is the PBH test so useful?

- No need to calculate the reachability matrix
- No need for explicit transformation into reachable/unreachable components
- We simply check where the reachability pencil loses rank

Summary

So what have we learnt about Reachability and controllability?

- Reachability and controllability are equivalent for continuous-time systems but differ in discrete time when A has zero eigenvalues.
- These properties can be tested numerically without explicit trajectory analysis.
- If a system is not reachable, it can be decomposed into reachable and unreachable components.
- The reachability pencil test helps identify the unreachable (hidden) modes of the system.

In the next lecture, we will look at analysing state-to-output interactions, following a similar approach.

Next Week

There will be a QMPlus quiz next week (Week 7 - Reflection Week) worth 20% of the module

- You will have a **90 minutes** to do the quiz but can take the quiz at any point in Week 7 (05/03-11/03)
- The quiz covers all the content from weeks 1-5 (i.e. the content from this lecture will not be in the quiz)

There will be no lecture or tutorial class next week!