Advanced Control Systems

Lecture 5: Reachability and Controllability

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Input-to-state properties

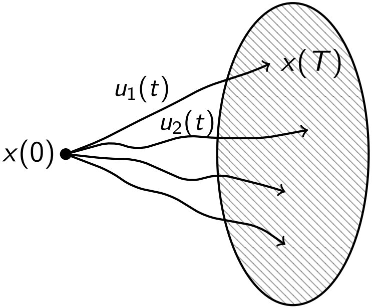
The goal is to describe the relationship between $u(t) \rightarrow x(t)$.

There are two perspectives to this problem:

- The state at $t = 0$ is given, and we would like to identify all states that can be **reached** in the future
- The state at $t = T$ is given, and we would like to identify all initial states $x(0)$ that can be **driven** to the final state

Reachability

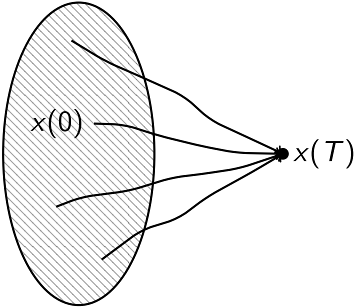
Reachability: Given $x(0)$, find all future states reachable by input sequences.



The set of states that can be reached at time T

Controllability

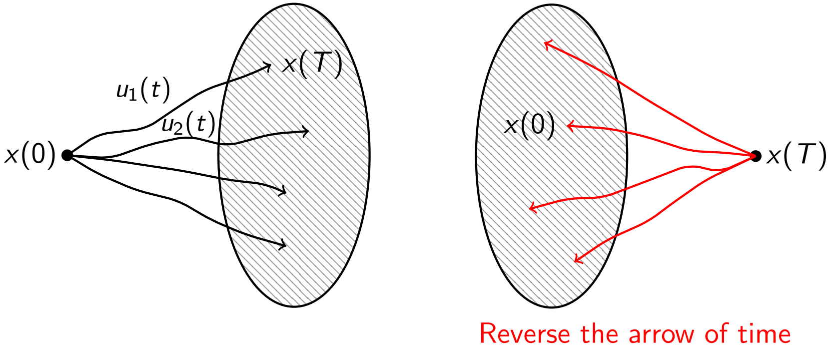
Controllability: Given $x(T)$, find all initial states x_0 that can be driven to $x(T)$.



The set of states that can
be steered/controlled to $x(T)$

Reachability vs Controllability

Observe what happens if we reverse the arrow of time:

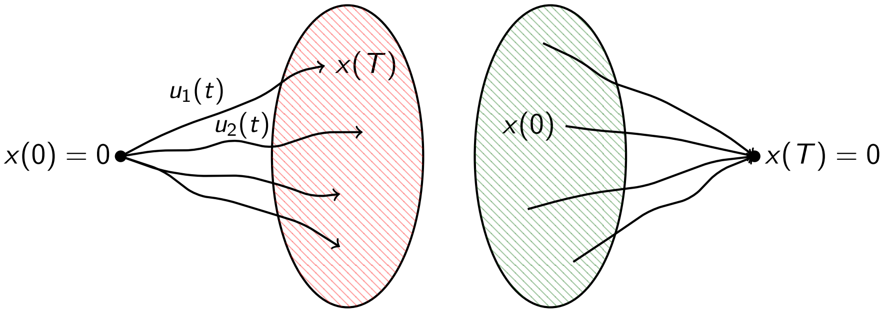


Reachability vs Controllability are equivalent in continuous-time systems but can differ in discrete-time systems.

Recall that continuous-time systems are always reversible, but discrete-time systems are not.

Linearity: Reachability vs Controllability

For linear systems, the **set of states** that can be **reached or controlled to** have a special structure: **linear subspaces of the state-space**.

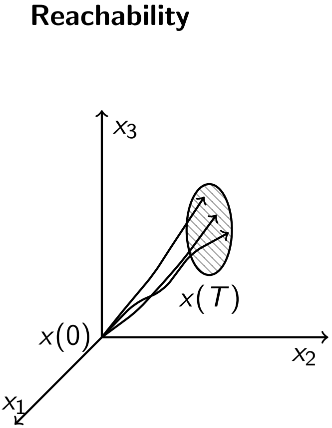


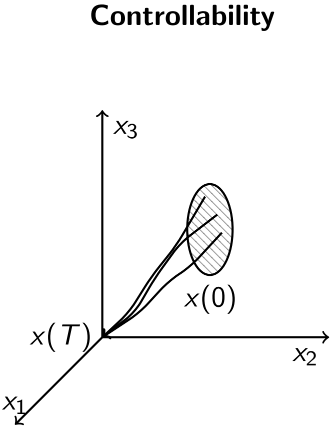
Linear subspace of the state-space

Linear subspace of the state-space

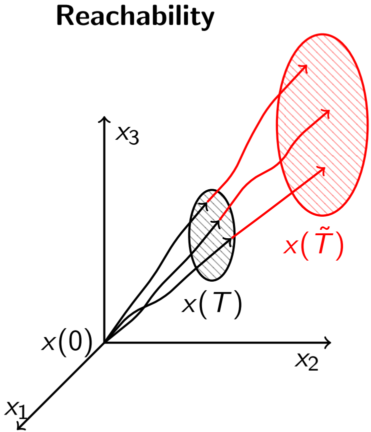
Often, for linear systems, we also simplify the analysis by setting $x(0) = 0$ (to the origin of the state space).

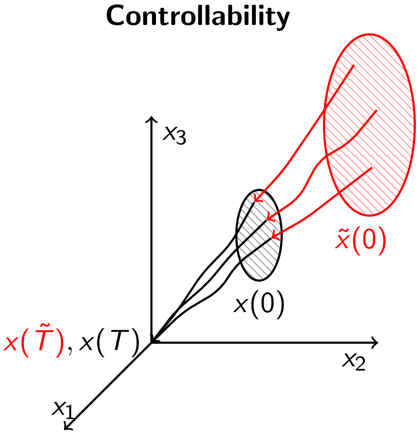
Phase Portrait: Reachability vs Controllability





Phase Portrait: Reachability vs Controllability





Reachability: Discrete-Time Systems

Let's start by looking at the reachability properties of a discrete-time system

$$x^+ = Ax + Bu.$$

- Reachability and controllability depend on A and B.

States that can be reached in one step are $x[1]$, and this can be written as:

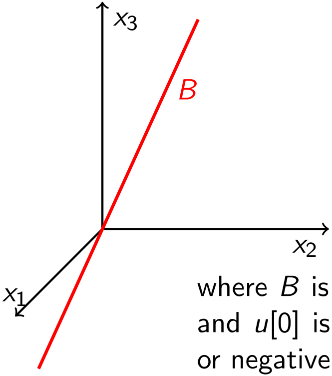
$$x[1] = Ax[0] + Bu[0] = Bu[0]\,, \quad (\text{since } x[0] = 0)$$

Reachability: Single-Input System - Step 1

What we would like to do is remove the effect of $u[0]$.

Let's consider a single-input system, i.e. $p = 1$ and $B = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$ is a vector.

$$x(1) = Bu[0]$$



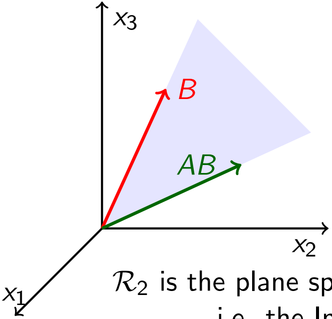
where B is the direction of travel and $u[0]$ is the distance (positive or negative) along the direction.

The set of points that can be reached in 1 step is linear space, $\mathrm{Im}(B) = \mathcal{R}_1$ (i.e. all the points on the red line).

Reachability: Single-Input System - Step 2

States that can be reached in two steps are $x[2]$, and this can be written as:

$$
\begin{aligned}
x[1] &= Ax[0] + Bu[0] = Bu[0] \\
x[2] &= Ax[1] + Bu[1] = Bu[1] + ABu[0] \\
&= \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} = \mathrm{Im}(\begin{bmatrix} B & AB \end{bmatrix}) = \mathcal{R}_2
\end{aligned}
$$



$\mathcal{R}_2$ is the plane spanned by A and AB.
i.e. the $\mathrm{Im}(\begin{bmatrix} B & AB \end{bmatrix})$

Reachability Subspaces

Notice that clearly, if a state $x[1]$ can be reached in one step, then it can also be reached in two steps.

This follows from the observation that we can take a step that effectively keeps us at zero (i.e., by choosing $u[0] = 0$) and then take a second step to reach $x[2]$.

Thus, every element of $\mathcal{R}_1$ is also in $\mathcal{R}_2$, implying:

$$\mathcal{R}_1 \subseteq \mathcal{R}_2$$

The reverse is not necessarily true. There clearly exist states that require exactly two steps to be reached but cannot be reached in a single step.

This means there exist elements in $\mathcal{R}_2$ that are not in $\mathcal{R}_1$.

Reachability: Single-Input System - Step 3

States that can be reached in three steps are $x[3]$, and this can be written as:

$$
\begin{aligned}
x[3] &= Ax[2] + Bu[2] \\
x[3] &= Bu[2] + A(Bu[1] + ABu[0]) = Bu[2] + ABu[1] + A^2Bu[0] \\
&= \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix} = \mathrm{Im}(\begin{bmatrix} B & AB & A^2B \end{bmatrix}) = \mathcal{R}_3
\end{aligned}
$$

Where

$$\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3$$

Reachability: Single-Input System - Step i

States that can be reached in three step are $x(i)$, and this can be written as:

$$\begin{aligned}
\mathcal{R}_1 &= \mathrm{Im}(B) \\
\mathcal{R}_2 &= \mathrm{Im}(\begin{bmatrix} B & AB \end{bmatrix}) \\
\mathcal{R}_3 &= \mathrm{Im}(\begin{bmatrix} B & AB & A^2B \end{bmatrix}) \\
\mathcal{R}_i &= \mathrm{Im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{i-1}B \end{bmatrix}
\end{aligned}$$

Where

$$\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \cdots \subseteq \mathcal{R}_i$$

Reachability: Step Termination

But when do we stop?

Cayley-Hamilton Theorem: if $p_\lambda(\lambda) = \det(\lambda I - A)$ then $p_\lambda(A) = 0$

Therefore is we have $\lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n = 0$ then

$$0 = A^n + \alpha_1 A^{n-1} + \cdots + I\alpha_n$$
$$-A^n = \alpha_1 A^{n-1} + \cdots + I\alpha_n$$

i.e. A^n can be written as a linear combination of lower powers of A.

Therefore if $i = n$ then

$$\mathcal{R}_n = \operatorname{Im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$
$$\mathcal{R}_{n+1} = \operatorname{Im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B & A^nB \end{bmatrix}$$

where

$$\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \cdots \subseteq \mathcal{R}_n = \mathcal{R}_{n+1}$$

Reachability Matrix

Given

$$x^+ = Ax + Bu.$$

$\mathcal{R}_1$: set of states reached in 1 step	$\mathcal{R}_1 = \mathrm{Im}(B)$
$\vdots$	$\vdots$
$\mathcal{R}_n$: set of states reached in n steps	$\mathcal{R}_n = \mathrm{Im}(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix})$

We call $R = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ the **reachability matrix** of the system.

System is reachable $\iff$ Any state can be reached in at most n steps.

This is equivalent to $\mathrm{Im}(R) =$ the state-space (i.e. rank $R = n$)

Reachability of Discrete-Time Systems

What about the dimensions of $R = \left[\underbrace{B}_{n\times p} \quad \underbrace{AB}_{n\times p} \quad \cdots \quad \underbrace{A^{n-1}B}_{n\times p}\right]$.

Notice if $p = 1$ then R is a **square matrix** and $m > 1$ then R is a wide matrix (i.e. has more columns than rows).

Therefore if $p = 1$ then the rank $R \iff \det(R) \neq 0$.

So, to compute the reachability of a single input discrete-time system, we just have to compute the determinate of R.

Computing the Reachability Matrix

There are two approaches when it comes to computing the reachability matrix:

$$R = \begin{bmatrix} B & AB & \underbrace{A^2B}_{A(AB) \text{ or } (A^2)B} & \underbrace{A^3B}_{A(A^2B) \text{ or } (A^3)B} & \cdots \end{bmatrix}$$

- $(A^2)B \rightarrow$ is a matrix $\times$ matrix $\times$ vector
- $A(AB) \rightarrow$ is a matrix $\times$ vector *(because we have computed AB)*

Likewise

- $(A^3)B \rightarrow$ is a matrix $\times$ matrix $\times$ matrix $\times$ vector
- $A(A^2B) \rightarrow$ is a matrix $\times$ vector *(because we have computed A^2B)*

So the second way is far more efficient!

Discrete-Time System Example

Consider the system:

$$x^+ = Ax + Bu$$

with matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The set of states that can be reached in 1 step is given by the image of B:

$$\mathcal{R}_1 = \mathrm{Im}(B) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Discrete-Time System Example

For two steps, we compute:

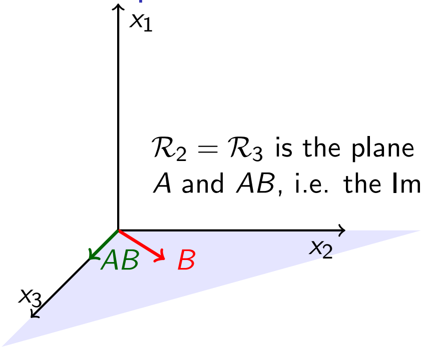
$$\mathcal{R}_2 = \mathrm{Im}(B, AB)$$

Performing the multiplication:

$$\mathcal{R}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \mathcal{R}_3, \quad \text{since } A(AB) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where this vector is already in $\mathcal{R}_2$, we conclude $\mathcal{R}_3 = \mathcal{R}_2$, meaning the reachable set has not expanded further.

Discrete-Time System Example: Geometric Interpretation



$\mathcal{R}_2 = \mathcal{R}_3$ is the plane spanned by A and AB, i.e. the $\mathrm{Im}(\begin{bmatrix} B & AB \end{bmatrix})$

This makes sense if you look back at

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

because

$$x_1^+ = 0$$

Discrete-Time System Example: Discussion

Imagine we change the system slightly

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

The reachability matrix does not change (the system is not reachable), but we now have dynamic behaviour in x_1 if we start in start at a non-zero x_1,

$$x_1^+ = 2x_1.$$

At each step, x_1 evolves independently (e.g., doubling at each step), but the **input signal does not influence this part of the system**.

Reachability in Continuous-Time Systems

Let's now characterise the reachability and controllability of linear continuous-time systems.

Recall: Continuous-time systems are reversible, implying equivalence between reachability and controllability.

Consider the system:

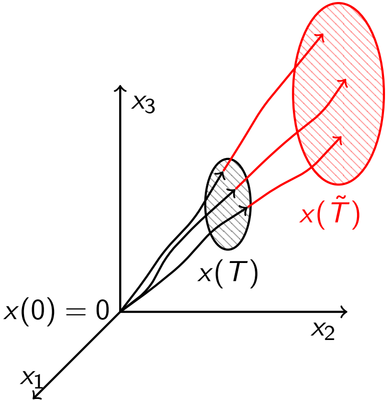
$$\dot{x} = Ax + Bu.$$

The set of reachable states at time T is:

$$x(T) = \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau, \quad \text{for} \quad u(t),\ t \in [0, T].$$

This equation is the **Lagrange formula** for state transition, where $x(0) = 0$, so only the forced response remains.

Phase Portrait of Reachability in Continuous-Time Systems



Note:

- We start at an initial state of zero, i.e., $x(0) = 0$
- Apply an input $u(t)$ to drive the state to a point at time T
- Different $u(t)$ yield different reachable states
- Continuous-time systems evolve over time intervals, not discrete steps

Reachability of Continuous vs Discrete Time Systems

The set of reachable states at time T is:

$$x(T) = \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau, \quad \text{for} \quad u(t),\ t \in [0, T]$$

- Unlike discrete-time systems (which use sums), continuous-time reachability involves an integral.
- The presence of the matrix exponential, $e^{A(T-\tau)}$, complicates analysis compared to discrete-time cases.

Our aim now is to capture the reachability properties in such a way that we are able to **remove time**.

Let's start by ignoring $u(t)$ and just focusing on $e^{A(T-\tau)}B$.

Simplifying the Reachability Expression

Recall that e^{At} by definition is:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^{n-1}t^{n-1}}{(n-1)!} + \frac{A^nt^n}{n!} + \frac{A^{n+1}t^{n+1}}{(n+1)!} + \ldots$$

However, using the Cayley-Hamilton theorem, we know that

$$A^n = -\alpha_0 I - \alpha_1 A - \alpha_2 A^2 \cdots - \alpha_{n-1}A^{n-1}.$$

Therefore

$$\begin{aligned} e^{At} = {} & I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^{n-1}t^{n-1}}{(n-1)!} + \\ & - \left[\alpha_0 I \frac{t^n}{n!} + \alpha_1 A \frac{t^n}{n!} + \alpha_2 A^2 \frac{t^n}{n!} + \cdots + \alpha_{n-1}A^{n-1}\frac{t^n}{n!}\right] \\ & - \left[\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\right] \\ & - \qquad \vdots \end{aligned}$$

Simplifying the Reachability Expression

If we gather terms together

$$
\begin{aligned}
e^{At} = {} & I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^{n-1}t^{n-1}}{(n-1)!} + \\
& - \left[\alpha_0 I \frac{t^n}{n!} + \alpha_1 A \frac{t^n}{n!} + \alpha_2 A^2 \frac{t^n}{n!} + \cdots + \alpha_{n-1} A^{n-1} \frac{t^n}{n!}\right] \\
& - \left[\cdots I \cdots A \cdots A^2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \right] \\
& \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

Rewriting the Exponential:

$$e^{At} = \varphi_0(t)I + \varphi_1(t)A + \cdots + \varphi_{n-1}(t)A^{n-1}$$

So we have now been able to go from the exponential, e^{At}, to an object that only contains powers of A from 0 to $n-1$.

Simplifying the Reachability Expression

So, the expression for the set of reachable states at time T is:

$$x(T) = \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau, \quad \text{for} \quad u(t),\ t \in [0, T]$$

But now using our simplification of e^{At}, we obtain:

$$x(T) = \int_0^T \left[\varphi_0(T-\tau)I + \varphi_1(T-\tau)A + \cdots + \varphi_{n-1}(T-\tau)A^{n-1}\right] Bu(\tau)\, d\tau$$

Notice that $\varphi_0(T-\tau)$, $\varphi_1(T-\tau)$, $\cdots$ $\varphi_{n-1}(T-\tau)$. are functions (they are not vectors), so we can write:

$$x(T) = \int_0^T \left[B\varphi_0(T-\tau)u(\tau) + AB\varphi_1(T-\tau)u(\tau) + \cdots + A^{n-1}B\varphi_{n-1}(T-\tau)u(\tau)\right] d\tau$$

Expression for Reachability

$$x(T) = \underbrace{\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}}_{\text{Reachability Matrix}} \begin{bmatrix} \int_0^T \varphi_0(T-\tau)u(\tau)\,d\tau \\ \int_0^T \varphi_1(T-\tau)u(\tau)\,d\tau \\ \vdots \\ \int_0^T \varphi_{n-1}(T-\tau)u(\tau)\,d\tau \end{bmatrix}$$

- The first term is nothing more than the reachability matrix:

$$R = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

- The second term in the expression captures the influence of the input $u(t)$.

So, we have been able to separate reachability analysis from explicit time dependence.

Expression for Reachability: Interpretation

This can be rewritten as:

$$x(T) = R \begin{bmatrix} \int \cdots \\ \int \cdots \\ \vdots \end{bmatrix} \iff x(T) \in \text{Im}(R)$$

where R is the reachability matrix

Therefore, if $x(T) \notin \text{Im}(R)$, then x(T) cannot be reached in time T

But R is independent of time, so we can actually say,
if $x(T) \notin \text{Im}(R)$, then x(T) cannot be reached in **any positive time**

However, we still do not know when $x(T) \in \text{Im}(R)$ if a $u(t)$ exists that is able to drive $x(0)$ to $X(T)$

Side Point: Continuous vs Discrete Time Systems

Notice for continuous-time systems, **time is essentially not important**.

If a state can be reached in one second, it can be reached in one millisecond, one hour, or one year, given the appropriate input.

In continuous-time systems, the reachability matrix R does not depend on time, which is a key difference from discrete-time systems.

Reachability Gramian

If $x(T) \in \text{Im}(R)$, we still need to verify whether we can assign the necessary linear combination of the columns of R.

We do this by defining the reachability Gramian:

$$W_t = \int_0^t \underbrace{e^{A(t-\tau)}}_{n\times n} \underbrace{B}_{n\times p} \underbrace{B'}_{p\times n} \underbrace{e^{A'(t-\tau)}}_{n\times n} d\tau$$

- W_t is a square matrix $n \times n$
- $W_0 = 0$
- $W_t = W_t'$ (Symmetric) and $W_t \geq 0$ (Positive semi-definite)
  - A matrix $M = M'$ is positive semi-definite if $x^T Mx \geq 0$, for all $x \in \mathbb{R}^n$
- $\text{Im}(W_t) = \text{Im}(R)$ for all $t > 0$
  - A non-trivial property that we won't prove
  - **Very important property as it relates a matrix that depends on time, W_t, to one that does not, R**

Selecting the Input Signal

Given:

$$x(T) = \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$

Choosing:

$$u(\tau) = B' e^{A'(T-\tau)} \beta$$

where β is a vector and a free parameter.

This leads to:

$$x(T) = \int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} \beta d\tau \iff x(T) = W_T \beta$$

Thus, X is a linear combination of the columns of W_T

If we pick $x(T) \in \mathrm{Im}(R)$ then $x(T) \in \mathrm{Im}(W_T)$

Reachability Condition

Therefore, given

$$\dot{x} = Ax + Bu$$

$$x(T) \in \text{Im}(R), \iff x(T) \text{ can be reached in any } T > 0$$

The system is **reachable** if all points can be reached for any $T > 0$

$$\text{Reachability} \iff \text{Im}(R) = \mathcal{R}^n \iff \text{rank}(R) = n \iff W_t > 0$$

- $W_t > 0 \iff W_t$ is invertible

Reachability Gramian and Invertibility

So we have picked a special selection of $u(t) = B' e^{A'(T-t)} \beta$ and shown

$$x(T) = W_T \beta$$

But if $W(T)$ is invertible then

$$\beta = W_T^{-1} x(T)$$

Therefore the input signal that drives $x(t)$ from 0 to $x(T)$ in time $T > 0$ is

$$u(t) = B' e^{A'(T-t)} W_T^{-1} x(T)$$

Note:

- For small T, W_T becomes small and its inverse, W_T^{-1} large
- Theoretically, we can steer from 0 to $\bar{x}$ in arbitrarily small time
- However, the required input energy increases as the interval shrinks

Continuous-Time System: Controllability vs Reachability

Consider the system:

$$\dot{x} = Ax + Bu, \quad x(0) \text{ is given}, \quad x(T) = 0$$

Applying Lagrange's equation:

$$\begin{aligned} 0 &= e^{AT}x(0) + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau \\ -e^{AT}x(0) &= \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau \end{aligned}$$

So, $x(0)$ is controllable to 0 in some time $T > 0$ if $-e^{AT}x(0)$ is reachable

Controllability Condition: Continuous-Time Systems

If $-e^{AT}x(0)$ is reachable then

$$e^{AT}x(0) \in \text{Im}(R) \iff x(0) \in e^{-AT}\text{Im}(R)$$

But e^{-AT} just contains elements such as $I, A, A^2, \cdots$ and R just contains elements such as $B, AB, A^2B, \cdots$

$$x(0) \in e^{-AT}\text{Im}(R) \iff x(0) \in \text{Im}(R)$$

So, $x(0)$ is controllable if $x(0) \in \text{Im}(R)$. Therefore, reachability and controllability are exactly the same for a continuous-time system.

Controllability Condition: Discrete-Time Systems

For discrete-time systems, e^{-AT} becomes $[A^{-1}]^k$, which is not invertible when you have eigenvalues at zero.

We have already seen that eigenvalues at zero for discrete-time systems have a very strange behaviour, such that you can reach zero in finite time.

A state $x[k]$ is controllable to zero in k steps if there exists an input sequence $u[0]$, $u[1]$, $\cdots$, $u[k-1]$ that drives the state from $x[k]$ to 0

$$0 = A^k x[k] + \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u[k-1] \\ u[k-2] \\ \vdots \\ u[0] \end{bmatrix}$$

Controllability Condition: Discrete-Time Systems

Equivalently

$$-A^k x[k] = \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u[k-1] \\ u[k-2] \\ \vdots \\ u[0] \end{bmatrix}$$

This implies that $x[k]$ is controllable if the state $-A^k x[k]$ is reachable in k steps, hence if

$$A^k x[k] \in \mathrm{Im}(R)$$

Therefore, a linear, discrete-time system is controllable in n steps if

$$\mathrm{Im}(A^n) \subseteq \mathrm{Im}(R)$$

Controllability Condition: Discrete-Time Systems

For discrete-time systems, the reachability and controllability properties are **NOT** equivalent:

1. Reachability $\implies$ controllability
   - Reachability implies $\mathrm{Im}(R) =$ the state-space from which it follows that $\mathrm{Im}(A^n) \subseteq \mathrm{Im}(R)$, so the system will be controllable.
2. Controllability $\not\Longrightarrow$ reachability
   - For example, $A = \mathbf{0}$ and rank $B < n$, then:
     rank $\begin{bmatrix} B & AB \cdots & A^{n-1}B \end{bmatrix} =$ rank $\begin{bmatrix} B & 0 & \cdots & 0 \end{bmatrix} < n$
     this system is not reachable even if it is controllable
3. If A is a full rank matrix, then the reachability and the controllability are the same for discrete-time systems.

Conclusion

- Reachability and controllability are fundamental properties in control theory
- The reachability Gramian plays a key role in system analysis
- Continuous-time systems: reachability $\iff$ controllability
- Discrete-time systems: reachability $\implies$ controllability
  but controllability $\not\Longrightarrow$ reachability
  (due to possible eigenvalues at zero)