Advanced Control Systems

Lecture 4: Lyapunov Stability

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Lyapunov Stability

Lyapunov Stability: Introduced in 1882 by Russian mathematician A.M. Lyapunov.

Other types of stability exist, such as **Lagrange stability**, but **Lyapunov stability** is the most widely used in applications.

Lyapunov Stability is crucial to studying the behaviour of a system's trajectories as time progresses, especially near equilibrium points.

Definition of Lyapunov Stability

Lyapunov Stability:

- Consider a system and an equilibrium point x_e.
- The equilibrium is stable if, for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that:

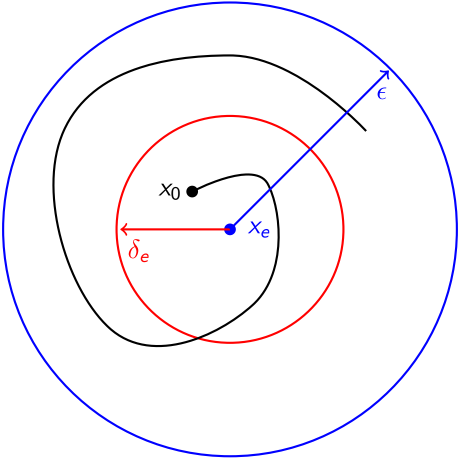
$$\|x(0) - x_e\| < \delta_\epsilon \implies \|x(t) - x_e\| < \epsilon \quad \text{for all } t \geq 0.$$

This means that if the initial perturbation is small, all future perturbations remain small.

Key Point

Lyapunov stability requires that for any small perturbations, ϵ, there exists a region around the equilibrium where the system remains within this region for all time.

Visualization of Stability



Asymptotic Stability

Asymptotic Stability:

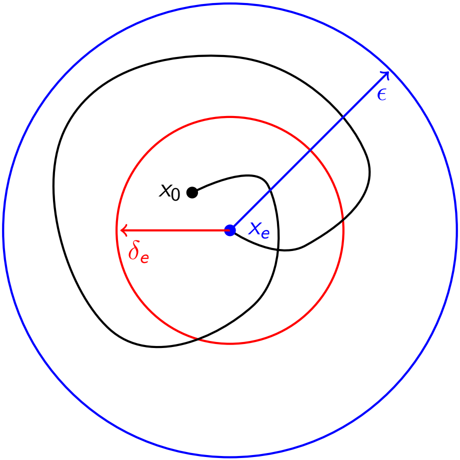
- The equilibrium is asymptotically stable if it is stable and there exists δ_ϵ such that:

$$\|x(0) - x_e\| < \delta_\epsilon \implies \lim_{t \to \infty} \|x(t) - x_e\| = 0.$$

Key Point

Asymptotic stability requires that all sufficiently small perturbations, ϵ, converge to the equilibrium.

Visualization of Asymptotic Stability



Unstable Equilibria

What does it mean for an equilibrium to be unstable?

- Small perturbations lead to large deviations.
- If trajectories move away from equilibrium as $t \to \infty$, the system is unstable.

Discrete-Time System Example

Suppose we have a linear discrete-time system given by:

$$x^+ = -x$$

Given an initial condition x_0, the system evolves as:

$$\begin{aligned} x[1] &= -x[0], \\ x[2] &= -x[1] = x[0], \\ x[3] &= -x[2] = -x[0], \\ x[4] &= x[0], \ldots \end{aligned}$$

Pattern:

$$x[k] = \begin{cases} -x[0], & \text{if } k \text{ is odd} \\ x[0], & \text{if } k \text{ is even} \end{cases}$$

Equilibrium of the System

An equilibrium is found by solving:

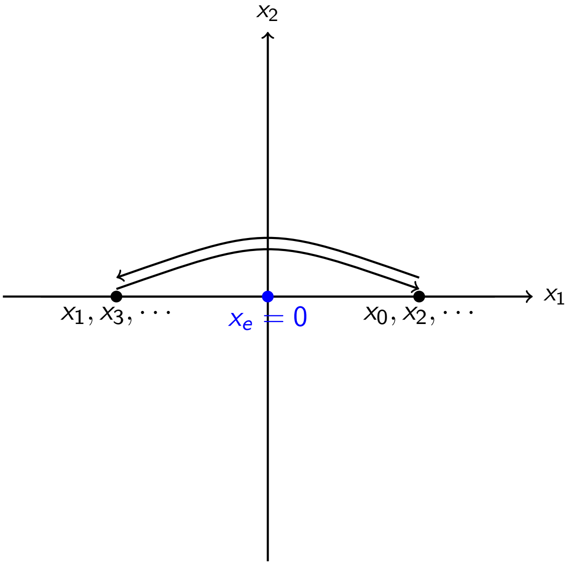
$$x^+ = x.$$

Substituting $x^+ = -x$:

$$x = -x \implies x = 0.$$

Unique equilibrium at $x = 0$.

Visualization of the Discrete-Time System



Stability Analysis

Stability condition:

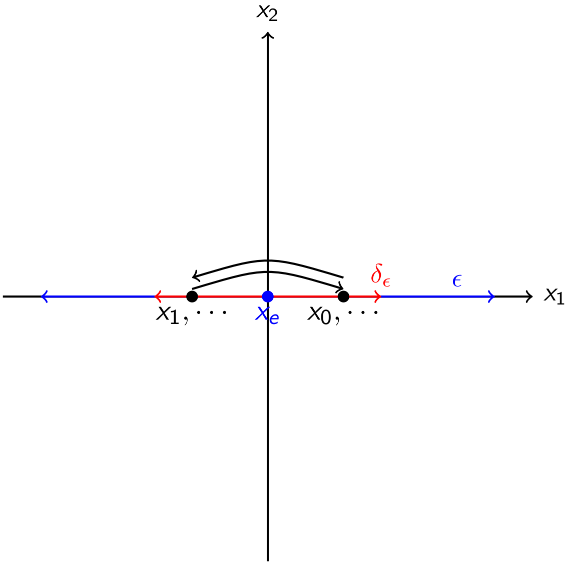
$$|x[0]| < \delta_\epsilon \implies |x[k]| < \epsilon, \quad \forall k \geq 0.$$

Since $|x[k]|$ remains constant, choosing:

$$|x[0]| < \delta_\epsilon \implies |x[0]| < \epsilon, \quad \forall k \geq 0.$$

Thus, the equilibrium $x_e = 0$ **is stable.**

Visualization of Discrete-Time System's Stability



Asymptotic Stability

The system does not converge to zero:

$$\lim_{k \to \infty} x[k] \neq 0.$$

The equilibrium is stable, but **not asymptotically stable**.

Eigenvalue Analysis

Rewriting the system:

$$x^+ = Ax, \quad \text{where } A = -1.$$

The eigenvalue of A:

$$\lambda_A = -1.$$

Eigenvalues will later be connected to stability analysis.

Continuous-Time System Example 1

Consider the system:

$$\dot{x}_1 = x_2, \tag{1}$$
$$\dot{x}_2 = -x_1. \tag{2}$$

The system matrix is:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix exponential is given by *(proof given in last lecture)*:

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

To find the equilibrium, solve:

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 0 \implies x_1 = 0, \quad x_2 = 0.$$

Thus, the equilibrium point is:

$$x_e = (0, 0).$$

Continuous-Time System Example 1

The system is:

$$\dot{x}_1 = x_2, \tag{3}$$
$$\dot{x}_2 = -x_1. \tag{4}$$

Multiply equations by x_1 and x_2:

$$x_1\dot{x}_1 = x_1 x_2, \tag{5}$$
$$x_2\dot{x}_2 = -x_1 x_2. \tag{6}$$

Adding them together gives you,

$$x_1\dot{x}_1 + x_2\dot{x}_2 = 0.$$

Continuous-Time System Example 1

Notice that the left-hand side is a derivative,

$$\frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right) = 0.$$

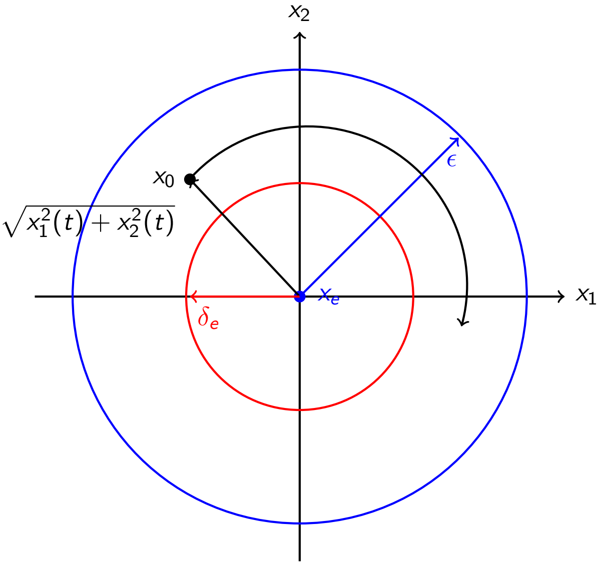
This implies:

$$x_1^2(t) + x_2^2(t) = \text{constant} = x_1^2(0) + x_2^2(0).$$

The state remains at a constant distance from the origin, meaning:

- Trajectories form circles centred at the origin.
- The system is **stable** but **not asymptotically stable**.

Visualisation of Example 1



Eigenvalue Analysis of Example 1

The characteristic polynomial of A is:

$$\det(\lambda I - A) = \lambda^2 + 1.$$

The eigenvalues are:

$$\lambda = \pm j.$$

Note that the eigenvalues have a modulus of 1.

Continuous-Time System Example 2

Let A be given by:

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

This gives the system:

$$\dot{x}_1 = x_2 - x_1, \tag{7}$$
$$\dot{x}_2 = -x_1 - x_2. \tag{8}$$

Multiply equations by x_1 and x_2:

$$x_1\dot{x}_1 = x_1x_2 - x_1^2, \tag{9}$$
$$x_2\dot{x}_2 = -x_1x_2 - x_2^2. \tag{10}$$

Adding them together gives you,

$$x_1\dot{x}_1 + x_2\dot{x}_2 = -(x_1^2 + x_2^2).$$

Continuous-Time System Example 2

Recognizing the left-hand side is a derivative,

$$x_1\dot{x}_1 + x_2\dot{x}_2 = \frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right) = -2\frac{(x_1^2 + x_2^2)}{2}.$$

Letting $r = \frac{x_1^2+x_2^2}{2}$, we get:

$$\underbrace{\frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right)}_{\dot{r}} = -2\underbrace{\frac{(x_1^2 + x_2^2)}{2}}_{r}.$$

Therefore

$$\dot{r} = -2r.$$

Solving this differential equation:

$$r(t) = r(0)e^{-2t}.$$

Proof

Step 1: Separate Variables

$$\frac{dr}{dt} = -2r \implies \frac{dr}{r} = -2dt$$

Step 2: Integrate Both Sides

$$\int \frac{dr}{r} = \int -2dt$$

$$\ln |r| = -2t + c$$

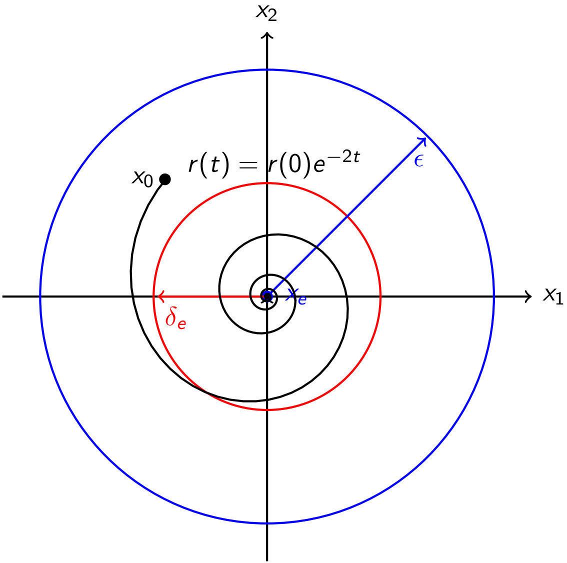
Step 3: Solve for r

$$|r| = e^{-2t+c} = e^c e^{-2t} = c' e^{-2t}$$

Step 4: Apply Initial Condition

$$r(0) = c' e^0 = c' \implies r(t) = r(0)e^{-2t}$$

Visualisation of Example 2



Eigenvalue Analysis of Example 2

Computing the eigenvalues of:

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The characteristic polynomial:

$$\det(\lambda I - A) = (\lambda + 1)^2 + 1 = \lambda^2 + 2\lambda + 2.$$

Solving,

$$\lambda = -1 \pm j.$$

The real part is negative!

Continuous-Time System Example 3

To analyze stability, consider the system:

$$\dot{x} = Ax, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{11}$$

We just need to modify the equations from example 2:

$$\dot{x}_1 = x_2 + x_1, \tag{12}$$
$$\dot{x}_2 = -x_1 + x_2. \tag{13}$$

Multiply equations by x_1 and x_2:

$$x_1\dot{x}_1 = x_1x_2 + x_1^2, \tag{14}$$
$$x_2\dot{x}_2 = -x_1x_2 + x_2^2. \tag{15}$$

Adding these:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2 + x_2^2.$$

Continuous-Time System Example 3

Again recognize the left-hand side as a time derivative:

$$\underbrace{\frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right)}_{\dot{r}} = x_1^2 + x_2^2 = 2\underbrace{\frac{(x_1^2 + x_2^2)}{2}}_{r}.$$

Letting $r = \frac{x_1^2+x_2^2}{2}$:

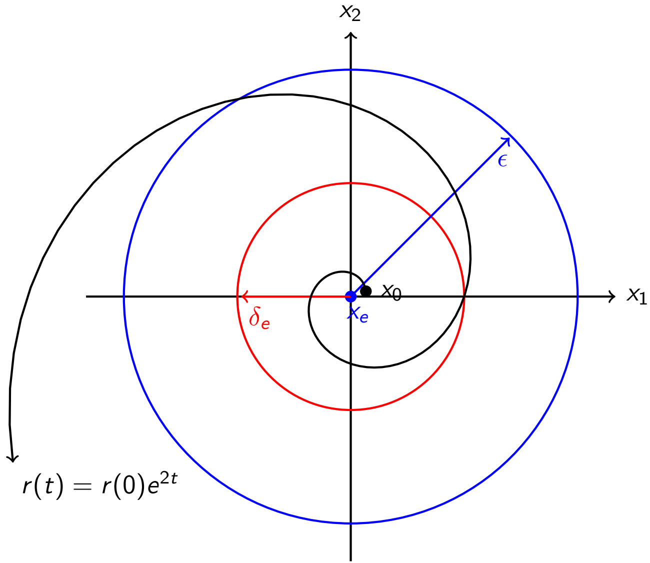
$$\dot{r} = 2r.$$

Solving:

$$r(t) = r(0)e^{2t}.$$

Since $r(t)$ increases exponentially, the equilibrium is **unstable**.

Visualisation of Example 3



Eigenvalues Analysis of Example 3

Computing eigenvalues of A:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

The characteristic polynomial:

$$\det(\lambda I - A) = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2.$$

Solving,

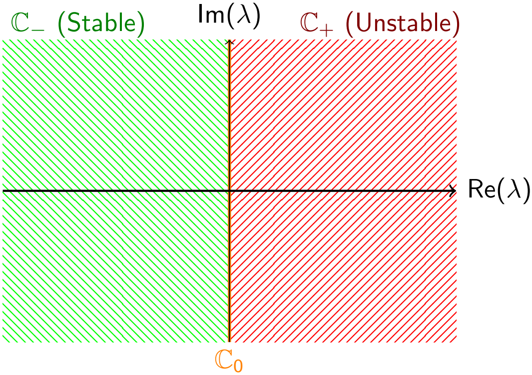
$$\lambda = 1 \pm j. \tag{16}$$

The real part is positive!

Notice that the exact value of the real part is not critical; what matters is its sign. A positive real part indicates an unstable system, while a negative real part ensures asymptotic stability.

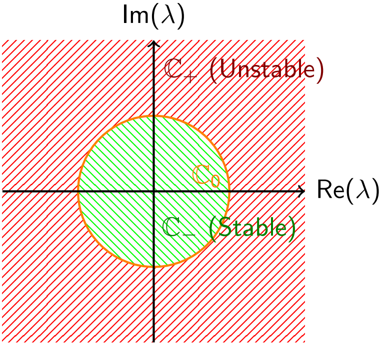
Classification in the Complex Plane

- **Asymptotically stable**: Eigenvalues in $\mathbb{C}_-$.
- **Unstable**: Eigenvalues in $\mathbb{C}_+$.
- **???**: Eigenvalues on $\mathbb{C}_0$ (next lecture).



Discrete-Time Stability Regions

- **Stable**: Inside the unit circle $\mathbb{C}_-$.
- **Unstable**: Outside the unit circle $\mathbb{C}_+$.
- **???**: Eigenvalues on $\mathbb{C}_0$ (next lecture).



Properties of Stability in Linear Systems

Key Properties:

- Stability can be assessed for linear systems without solving for trajectories.
- Stability of one trajectory implies stability of all trajectories.
- Asymptotic stability at the origin ($x_e = 0$) implies:
  - The origin is the only equilibrium for $u = 0$.
  - Asymptotic stability is global.
- Stability is inherited by any representation related through valid coordinate transformations.
  - Stability is preserved if L is invertible.

Homework

Consider the discrete-time system:

$$x^+ = \alpha x.$$

Prove:

- $|\alpha| < 1 \Rightarrow x_e$ is asymptotically stable.
- $|\alpha| = 1 \Rightarrow x_e$ is stable but not asymptotically stable.
- $|\alpha| > 1 \Rightarrow x_e$ is unstable.