# Advanced Control Systems

## Lecture 4: Lyapunov Stability

Aidan O. T. Hogg

EECS, Queen Mary University of London

a.hogg@qmul.ac.uk

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# Lyapunov Stability

**Lyapunov Stability:** Introduced in 1882 by Russian mathematician A.M. Lyapunov.

Other types of stability exist, such as **Lagrange stability**, but **Lyapunov stability** is the most widely used in applications.

**Lyapunov Stability** is crucial to studying the behaviour of a system's trajectories as time progresses, especially near equilibrium points.

# Definition of Lyapunov Stability

**Lyapunov Stability:**

- Consider a system and an equilibrium point $x_e$.
- The equilibrium is stable if, for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that:

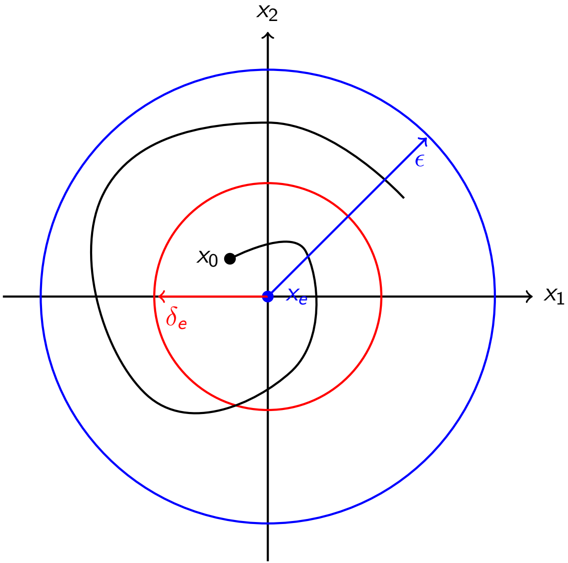
$$\|x(0) - x_e\| < \delta_\epsilon \implies \|x(t) - x_e\| < \epsilon \quad \text{for all } t \geq 0.$$

This means that if the initial perturbation is small, all future perturbations remain small.

**Key Point**

Lyapunov stability requires that for any small perturbations, $\epsilon$, there exists a region around the equilibrium where the system remains within this region for all time.

# Phase Portrait of Stability



# Asymptotic Stability

**Asymptotic Stability:**

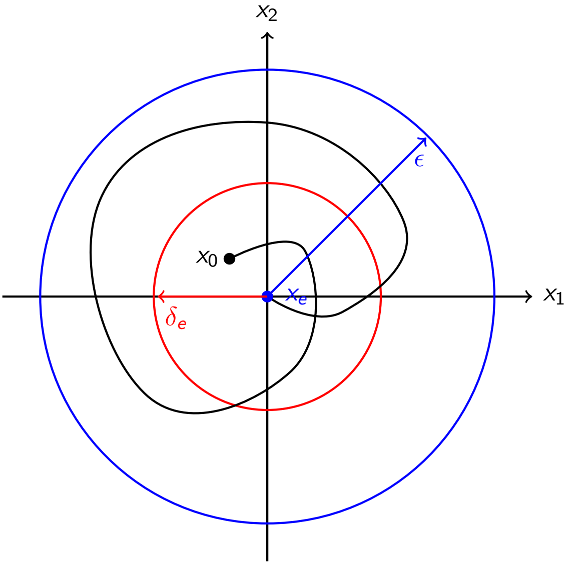
- The equilibrium is asymptotically stable if it is stable and there exists $\delta_\epsilon$ such that:

$$\|x(0) - x_e\| < \delta_\epsilon \implies \lim_{t\to\infty} \|x(t) - x_e\| = 0.$$

**Key Point**

Asymptotic stability requires that all sufficiently small perturbations, $\epsilon$, converge to the equilibrium.

# Phase Portrait of Asymptotic Stability



# Unstable Equilibria

What does it mean for an equilibrium to be unstable?

- Small perturbations lead to large deviations.
- If trajectories move away from equilibrium as $t \to \infty$, the system is unstable.

# Discrete-Time System Example

Suppose we have a linear discrete-time system given by:

$$x^+ = -x$$

Given an initial condition $x_0$, the system evolves as:

$$\begin{aligned}
x[1] &= -x[0], \\
x[2] &= -x[1] = x[0], \\
x[3] &= -x[2] = -x[0], \\
x[4] &= x[0], \ldots
\end{aligned}$$

Pattern:

$$x[k] = \begin{cases} -x[0], & \text{if } k \text{ is odd} \\ x[0], & \text{if } k \text{ is even} \end{cases}$$

# Equilibrium of the System

An equilibrium is found by solving:

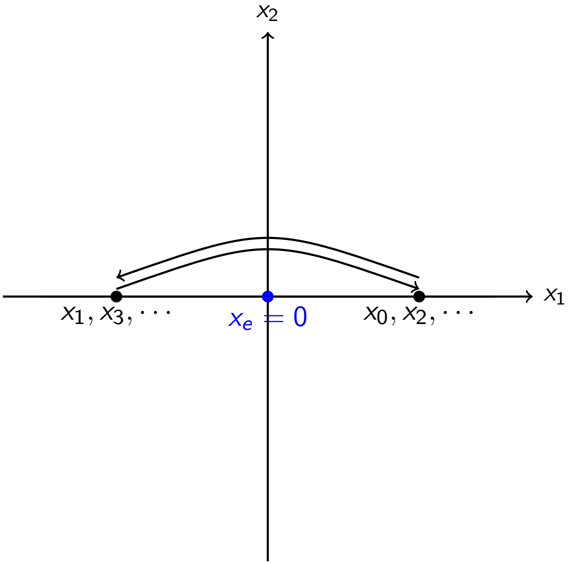
$$x^+ = x.$$

Substituting $x^+ = -x$:

$$x = -x \implies x = 0.$$

**Unique equilibrium at** $x = 0$.

# Phase Portrait of the Discrete-Time System



# Stability Analysis

Stability condition:

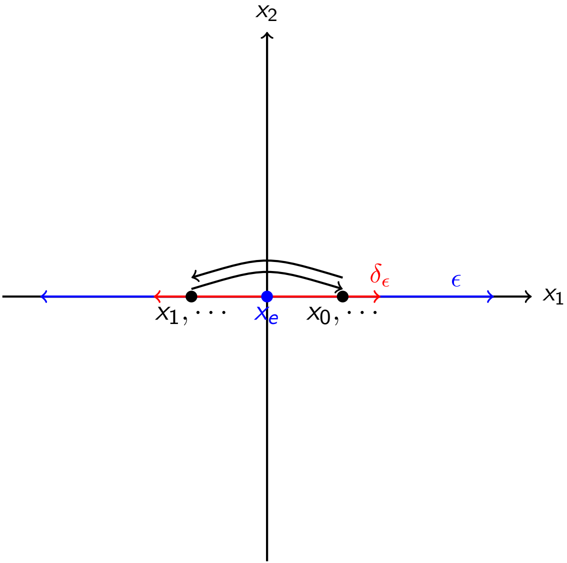
$$|x[0]| < \delta_\epsilon \implies |x[k]| < \epsilon, \quad \forall k \geq 0.$$

Since $|x[k]|$ remains constant, choosing:

$$|x[0]| < \delta_\epsilon \implies |x[0]| < \epsilon, \quad \forall k \geq 0.$$

Thus, the equilibrium $x_e = 0$ **is stable.**

# Phase Portrait of Discrete-Time System's Stability



# Asymptotic Stability

The system does not converge to zero:

$$\lim_{k\to\infty} x[k] \neq 0.$$

The equilibrium is stable, but **not asymptotically stable**.

# Eigenvalue Analysis

Rewriting the system:

$$x^+ = Ax, \quad \text{where } A = -1.$$

The eigenvalue of $A$:

$$\lambda_A = -1.$$

**Eigenvalues will later be connected to stability analysis.**

# Continuous-Time System Example 1

Consider the system:

$$\dot{x}_1 = x_2, \tag{1}$$
$$\dot{x}_2 = -x_1. \tag{2}$$

The system matrix is:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix exponential is given by *(proof given in last lecture)*:

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

To find the equilibrium, solve:

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 0 \implies x_1 = 0, \quad x_2 = 0.$$

Thus, the equilibrium point is:

$$x_e = (0, 0).$$

# Continuous-Time System Example 1

The system is:

$$\dot{x}_1 = x_2, \tag{3}$$
$$\dot{x}_2 = -x_1. \tag{4}$$

Multiply equations by $x_1$ and $x_2$:

$$x_1\dot{x}_1 = x_1 x_2, \tag{5}$$
$$x_2\dot{x}_2 = -x_1 x_2. \tag{6}$$

Adding them together gives you,

$$x_1\dot{x}_1 + x_2\dot{x}_2 = 0.$$

# Continuous-Time System Example 1

Notice that the left-hand side is a derivative,

$$\frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right) = 0.$$

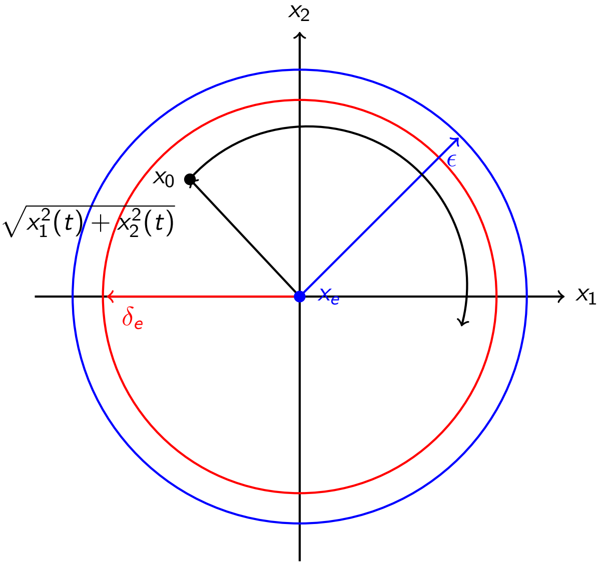
This implies:

$$x_1^2(t) + x_2^2(t) = \text{constant} = x_1^2(0) + x_2^2(0).$$

The state remains at a constant distance from the origin, meaning:

- Trajectories form circles centred at the origin.
- The system is **stable** but **not asymptotically stable**.

# Phase Portrait of Example 1



# Eigenvalue Analysis of Example 1

The characteristic polynomial of $A$ is:

$$\det(\lambda I - A) = \lambda^2 + 1.$$

The eigenvalues are:

$$\lambda = \pm j.$$

Note that the eigenvalues have a modulus of 1.

# Continuous-Time System Example 2

Let $A$ be given by:

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

This gives the system:

$$\dot{x}_1 = x_2 - x_1, \tag{7}$$
$$\dot{x}_2 = -x_1 - x_2. \tag{8}$$

Multiply equations by $x_1$ and $x_2$:

$$x_1\dot{x}_1 = x_1 x_2 - x_1^2, \tag{9}$$
$$x_2\dot{x}_2 = -x_1 x_2 - x_2^2. \tag{10}$$

Adding them together gives you,

$$x_1\dot{x}_1 + x_2\dot{x}_2 = -(x_1^2 + x_2^2).$$

# Continuous-Time System Example 2

Recognizing the left-hand side is a derivative,

$$x_1\dot{x}_1 + x_2\dot{x}_2 = \frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right) = -2\frac{(x_1^2 + x_2^2)}{2}.$$

Letting $r = \frac{x_1^2+x_2^2}{2}$, we get:

$$\underbrace{\frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right)}_{\dot{r}} = -2\underbrace{\frac{(x_1^2 + x_2^2)}{2}}_{r}.$$

Therefore

$$\dot{r} = -2r.$$

Solving this differential equation:

$$r(t) = r(0)e^{-2t}.$$

# Proof

Step 1: Separate Variables

$$\frac{dr}{dt} = -2r \implies \frac{dr}{r} = -2dt$$

Step 2: Integrate Both Sides

$$\int \frac{dr}{r} = \int -2dt$$

$$\ln |r| = -2t + c$$

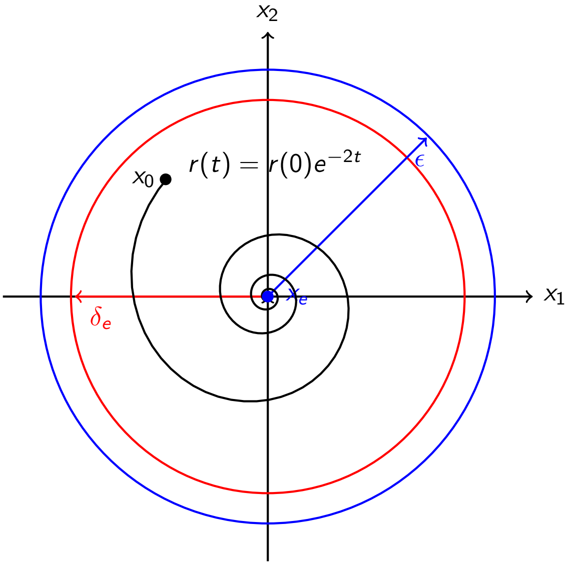
Step 3: Solve for $r$

$$|r| = e^{-2t+c} = e^c e^{-2t} = c' e^{-2t}$$

Step 4: Apply Initial Condition

$$r(0) = c' e^0 = c' \implies r(t) = r(0)e^{-2t}$$

# Phase Portrait of Example 2



# Eigenvalue Analysis of Example 2

Computing the eigenvalues of:

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The characteristic polynomial:

$$\det(\lambda I - A) = (\lambda + 1)^2 + 1 = \lambda^2 + 2\lambda + 2.$$

Solving,

$$\lambda = -1 \pm j.$$

**The real part is negative!**

# Continuous-Time System Example 3

To analyze stability, consider the system:

$$\dot{x} = Ax, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{11}$$

We just need to modify the equations from example 2:

$$\dot{x}_1 = x_2 + x_1, \tag{12}$$
$$\dot{x}_2 = -x_1 + x_2. \tag{13}$$

Multiply equations by $x_1$ and $x_2$:

$$x_1\dot{x}_1 = x_1 x_2 + x_1^2, \tag{14}$$
$$x_2\dot{x}_2 = -x_1 x_2 + x_2^2. \tag{15}$$

Adding these:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2 + x_2^2.$$

# Continuous-Time System Example 3

Again recognize the left-hand side as a time derivative:

$$\underbrace{\frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right)}_{\dot{r}} = x_1^2 + x_2^2 = 2\,\underbrace{\frac{(x_1^2 + x_2^2)}{2}}_{r}.$$

Letting $r = \frac{x_1^2+x_2^2}{2}$:

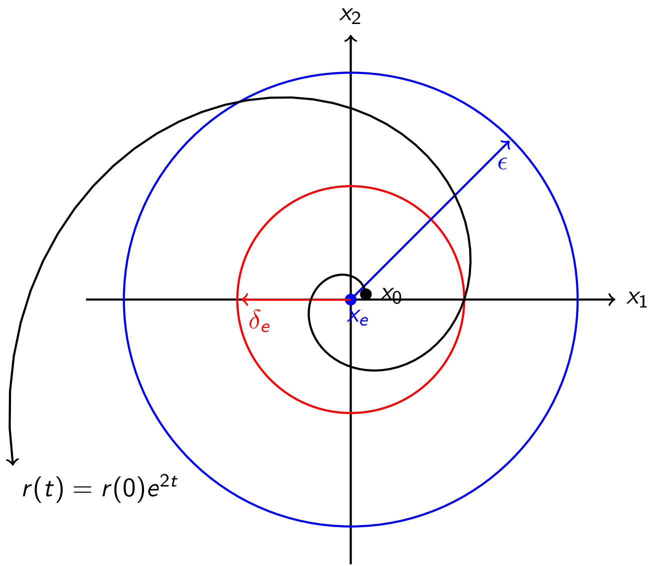
$$\dot{r} = 2r.$$

Solving:

$$r(t) = r(0)e^{2t}.$$

Since $r(t)$ increases exponentially, the equilibrium is **unstable**.

# Phase Portrait of Example 3



# Eigenvalues Analysis of Example 3

Computing eigenvalues of $A$:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

The characteristic polynomial:

$$\det(\lambda I - A) = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2.$$

Solving,

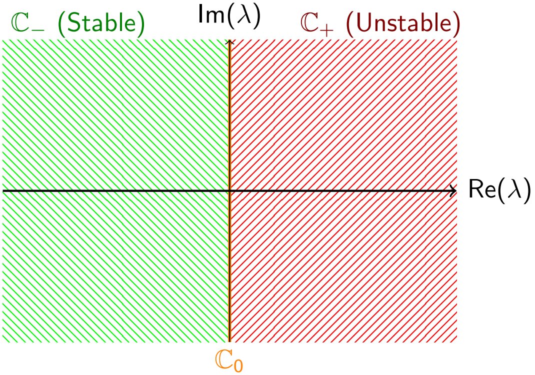
$$\lambda = 1 \pm j. \tag{16}$$

**The real part is positive!**

*Notice that the exact value of the real part is not critical; what matters is its sign. A positive real part indicates an unstable system, while a negative real part ensures asymptotic stability.*

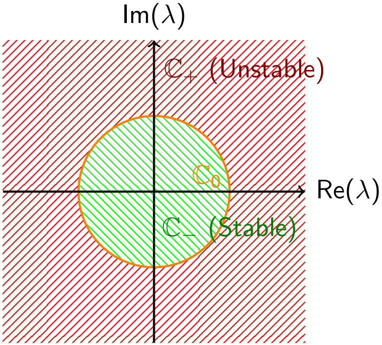
# Classification in the Complex Plane

- **Asymptotically stable**: Eigenvalues in $\mathbb{C}_-$.
- **Unstable**: Eigenvalues in $\mathbb{C}_+$.
- **???**: Eigenvalues on $\mathbb{C}_0$ (next lecture).



# Discrete-Time Stability Regions

- **Stable**: Inside the unit circle $\mathbb{C}_-$.
- **Unstable**: Outside the unit circle $\mathbb{C}_+$.
- **???**: Eigenvalues on $\mathbb{C}_0$ (next lecture).



# Properties of Stability in Linear Systems

**Key Properties:**

- Stability can be assessed for linear systems without solving for trajectories.
- Stability of one trajectory implies stability of all trajectories.
- Asymptotic stability at the origin ($x_e = 0$) implies:
  - The origin is the only equilibrium for $u = 0$.
  - Asymptotic stability is global.
- Stability is inherited by any representation related through valid coordinate transformations.
  - Stability is preserved if $L$ is invertible.

# Homework Exercise

Consider the discrete-time system:

$$x^+ = \alpha x.$$

Prove:

- $|\alpha| < 1 \Rightarrow x_e$ is asymptotically stable.
- $|\alpha| = 1 \Rightarrow x_e$ is stable but not asymptotically stable.
- $|\alpha| > 1 \Rightarrow x_e$ is unstable.