Advanced Control Systems

Lecture 3: Trajectories Cont. and Coordinate Transformations

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Discrete-Time Linear System

Consider a discrete-time linear system described by:

$$
\begin{aligned}
x[k+1] &= Ax[k] + Bu[k], \\
y[k] &= Cx[k] + Du[k].
\end{aligned}
$$

The first equation describes the dynamic behaviour, while the second maps the state and input to the output.

Problem Statement

Given an initial state $x[0]$ and a sequence of input values $u[0], u[1], \ldots$

We want to compute:

- The state $x[k]$ for all $k \geq 0$.
- The output $y[k]$ for all $k \geq 0$.

Discrete-Time Linear System

Consider a linear, discrete-time system:

$$x[k+1] = Ax[k] + Bu[k]. \quad (1)$$

Expanding for multiple time steps:

$$x[1] = Ax[0] + Bu[0],$$

$$\begin{aligned} x[2] &= Ax[1] + Bu[1], \\ x[2] &= A(Ax[0] + Bu[0]) + Bu[1], \\ x[2] &= A^2x[0] + ABu[0] + Bu[1], \\ x[2] &= A^2x[0] + [AB, B]\begin{bmatrix} u[0] \\ u[1] \end{bmatrix}. \end{aligned}$$

Discrete-Time Linear System

$$x[2] = A^2 x[0] + [AB, B] \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}.$$

$$\begin{aligned} x[3] &= Ax[2] + Bu[2], \\ x[3] &= A\left(A^2 x[0] + [AB, B] \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}\right) + Bu[2], \\ x[3] &= A^3 x[0] + [A^2 B, AB, B] \begin{bmatrix} u[0] \\ u[1] \\ u[2] \end{bmatrix}. \end{aligned}$$

Discrete-Time Linear System

The general form:

$$x[k] = \underbrace{A^k x[0]}_{\text{free response}} + \underbrace{[A^{k-1}B, \cdots AB, B] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-1] \end{bmatrix}}_{\text{forced response}}.$$

$$x[k] = \underbrace{A^k x[0]}_{\text{free response}} + \underbrace{\sum_{i=0}^{k-1} A^{k-1-i} B u[i]}_{\text{forced response}}.$$

Free and Forced Response of the Output

Consider the output equation of a discrete-time system:

$$x[k+1] = Ax[k] + Bu[k], \quad y[k] = Cx[k] + Du[k]$$

The output response is:

$$y[k] = \underbrace{CA^k x[0]}_{\text{free response}} + \underbrace{[CA^{k-1}, \cdots CAB, CB] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-1] \end{bmatrix} + Du[k]}_{\text{forced response}}.$$

$$y[k] = \underbrace{CA^k x_0}_{\text{free response}} + \underbrace{\sum_{i=0}^{k-1} CA^{k-1-i} Bu[i] + Du[k]}_{\text{forced response}},$$

Discrete vs. Continuous Systems

$$x[k] = \underbrace{A^k x[0]}_{\text{free response}} + \underbrace{\sum_{i=0}^{k-1} A^{k-1-i} Bu[i]}_{\text{forced response}}.$$

$$x(t) = \underbrace{e^{At} x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau}_{\text{forced response}}$$

Traveling Backward in Time

For continuous-time systems:

$$x[0] = e^{-At}x(t) - e^{-At}\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau. \tag{2}$$

Continuous-time systems are always reversible recall $(e^{At})^{-1} = e^{-At}$.

For discrete-time systems:

$$x[0] = A^{-k}x[k] - \sum_{i=0}^{k-1} A^{-i-1}Bu[i]. \tag{3}$$

Reversibility condition for discrete-time systems: $\det(A) \neq 0$ (i.e. A does not have an eigenvalue at 0).

$$x[0] = [A^{-1}]^k x[k] - [A^{-1}]^k \sum_{i=0}^{k-1} A^{k-i-1}Bu[i]. \tag{4}$$

Discrete-Time Systems Examples

Example 1: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- Compute $A^2 = \mathbf{0}$, so the $A^3 = \mathbf{0}$, $A^4 = \mathbf{0}$ etc.

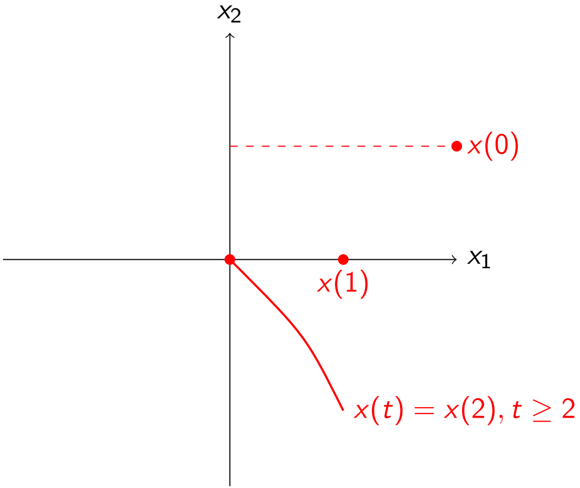
If $x[k+1] = Ax[k]$ then given x[0]:

$$x[1] = Ax[1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \begin{bmatrix} x_2[0] \\ 0 \end{bmatrix}; \qquad x[2] = A^2 x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

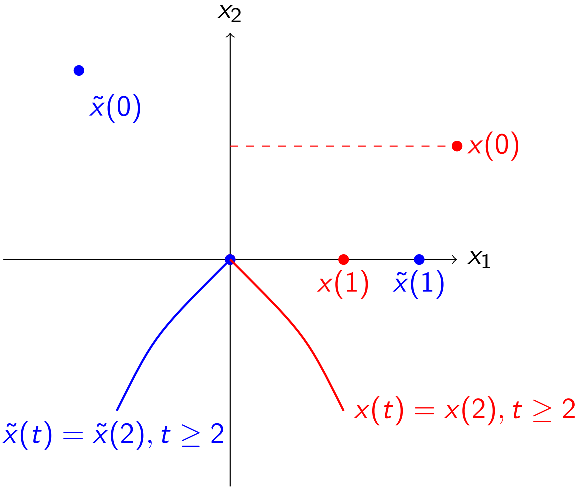
This means that any state goes to the origin in at most two steps. Such systems are not reversible: multiple initial conditions can lead to the same state, making it impossible to determine a unique past state.

This is a unique property of discrete-time systems as continuous-time systems cannot reach zero in finite time but only asymptotically.

Discrete-Time Systems Examples



Discrete-Time Systems Examples



Discrete-Time Systems Examples

Example 2: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A^0 = I, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Therefore A^t is:

$$A^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Coordinate Transformations: Continuous-Time

The choice of state variables in a system is not unique, and we can apply transformations without losing information. Consider the continuous-time system:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du. \tag{5}$$

We define the transformation as:

$$x = L\hat{x} \implies \hat{x} = L^{-1}x \text{ iff } \det(L) \neq 0. \tag{6}$$

Taking the time derivative:

$$\dot{\hat{x}} = L^{-1}\dot{x}. \tag{7}$$

Substituting $\dot{x} = Ax + Bu$:

$$\dot{\hat{x}} = L^{-1}Ax + L^{-1}Bu. \tag{8}$$

Using $x = L\hat{x}$, we obtain:

$$\dot{\hat{x}} = L^{-1}AL\hat{x} + L^{-1}Bu. \tag{9}$$

Transformed System Matrices

Defining:

$$\hat{A} = L^{-1}AL, \quad \hat{B} = L^{-1}B, \tag{10}$$

the system becomes:

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u. \tag{11}$$

Similarly, for the output equation:

$$y = Cx + Du \implies y = CL\hat{x} + Du. \tag{12}$$

Defining $\hat{C} = CL$ and $\hat{D} = D$, we obtain:

$$y = \hat{C}\hat{x} + \hat{D}u. \tag{13}$$

Eigenvalues and Similarity

A and $\hat{A}$ are similar matrices:

$$\hat{A} = L^{-1}AL. \tag{14}$$

i.e. they share the same eigenvalues. Consider the following

$$\det(\lambda I - \hat{A}) = \det(\lambda L^{-1}L - L^{-1}AL). \tag{15}$$

Using the determinant property $\det(AB) = \det(A)\det(B)$, we get:

$$\det(\lambda I - \hat{A}) = \det(L^{-1})\det(\lambda I - A)\det(L). \tag{16}$$

Since $\det(L^{-1})\det(L) = 1$, we conclude:

$$\det(\lambda I - \hat{A}) = \det(\lambda I - A). \tag{17}$$

Therefore, the eigenvalues of A and $\hat{A}$ are identical, meaning coordinate transformations do not affect the dynamical properties of a system.

Coordinate Transformations: Discrete-Time

The same is true for discrete-time system:

$$x[k+1] = Ax[k] + Bu[k], \quad y[k] = Cx[k] + Du[k], \tag{18}$$

applying $x = L\hat{x}$, we obtain:

$$\hat{x}[k+1] = \hat{A}\hat{x}[k] + \hat{B}u[k], \quad y[k] = \hat{C}\hat{x}[k] + Du[k]. \tag{19}$$

where:

$$\hat{A} = L^{-1}AL, \quad \hat{B} = L^{-1}B, \quad \hat{C} = CL. \tag{20}$$

Key Takeaway

Coordinate transformations preserve system structure and eigenvalues, ensuring that fundamental dynamical properties remain unchanged.