Lecture Exercises (part 2)

Question 1. *Consider the following discrete-time system*

$$x^+ = Ax + Bu$$

where

$$A = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{for } -1 \leq \alpha \leq 1$$

(a) Determine the reachability and controllability as a function of α.

(b) Study stability properties as a function of α.

(c) Design a state feedback control law such that the closed-loop system is asymptotically stable.

Solution 1.

(a)
$$R = [B \quad AB] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where the rank $= 2$, $\det = 1 \Rightarrow$ **the system is reachable**.

We also know that reachability $\Rightarrow$ controllability, so no need to compute A^2. We know **the system is controllable** because it is reachable.

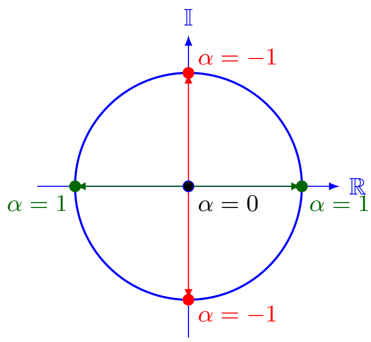
(b) To determine if the system is stable in terms of Lyapunov stability, we have to compute the following
$$P(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda & -\alpha \\ -1 & \lambda \end{bmatrix} = \lambda^2 - \alpha = 0$$

This gives the following eigenvalues in terms of α

$$\lambda_{1,2} = \pm\sqrt{\alpha}$$

Note that if

- $\alpha > 0$ then $\lambda = \pm\sqrt{\alpha} \implies$ real roots
- $\alpha < 0$ then $\lambda = \pm\sqrt{\alpha} \implies$ imaginary roots
- $\alpha = 0$ then $\lambda = 0$



Therefore, in terms of Lyapunov stability:

- The system is asymptotically stable for $\alpha \in (0, 1)$ [both roots are inside the unit circle]
- The system is asymptotically stable for $\alpha \in (-1, 0)$ [both roots are inside the unit circle]
- The system is stable for $\alpha = 1$ [both roots are simple roots (geometric multiplicity = 1) on the unit circle]
- The system is stable for $\alpha = -1$ [both roots are simple roots (geometric multiplicity = 1) on the unit circle]

To find a $A + BK$ such that the closed-loop system is asymptotically stable, we first need to compute the characteristic equation

$$P_\lambda(A + BK) = \mathbf{0}$$

To do this, let's first compute

$$A + BK = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} K_1 & \alpha + K_2 \\ 1 & 0 \end{bmatrix}$$

$$P_\lambda(A + BK) = \det(\lambda I - (A + BK)) = 0$$

$$\det\left(\begin{bmatrix} \lambda - K_1 & -\alpha - K_2 \\ -1 & \lambda \end{bmatrix}\right) = \lambda(\lambda - K_1) - (\alpha + K_2) = \lambda^2 - \lambda K_1 - (\alpha + K_2)$$

If we want all eigenvalues to be at zero, then we want the characteristic equation of the closed-loop system to be equal to λ^2. Therefore

$$\lambda^2 - \lambda K_1 - (\alpha + K_2) = \lambda^2$$

This implies that

$$K_1 = 0, \qquad K_2 = -\alpha$$

Note that if $\alpha = 0$, then there is no feedback at all, which may seem odd, but if we examine A, then we see that when $\alpha = 0$

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and the eigenvalues are already at zero (so no feedback is needed).

Question 2. *Consider the following continuous-time system*

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

(a) *Study the stability properties as a function of* α.

(b) *Show that the system is non-controllable but is stabilizable.*

(c) *Design a feedback controller that renders the closed-loop system asymptotically stable for all values of* α.

Solution 2.

(a) Note the the matrix A is upper triangular and, therefore,

$$\lambda(A) = \{-3, -2, \alpha\}$$

Since λ_1 and λ_2 lie in the left-hand side plane (i.e. -3 and -2 are always in $\mathbb{C}^-$), for the system to be asymptotically stable, α needs to also lie in the left-hand side plane. Therefore if

- $\alpha < 0$: Asymptotically Stable
- $\alpha = 0$: Stable [note that the geometric multiplicity of $\lambda = \alpha = 0$ is 1]
- $\alpha > 0$: Unstable

(b) To show that the system is non-controllable but still stabilizable, we need to prove that all unreachable modes are in the left-hand side plane.

Therefore, we first need to compute the reachability pencil

$$[\lambda I - A \mid B] = \left[\begin{array}{ccc|c} \lambda + 3 & -1 & 0 & 0 \\ 0 & \lambda + 2 & 0 & 0 \\ 0 & 0 & \lambda - \alpha & 1 \end{array}\right]$$

Now, we need to assess for which eigenvalues the reachability pencil loses rank. Since the last column is all zero and one element of 1, we know that

$$\text{rank}\,[\lambda I - A \mid B] = 1 + \text{rank}\begin{bmatrix} \lambda + 3 & -1 & 0 \\ 0 & \lambda + 2 & 0 \end{bmatrix}$$

If $\lambda = -3$ then

$$\text{rank}\begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = 1$$

So the overall rank is $2 < n$.

If $\lambda = -2$ then

$$\text{rank}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1$$

Again, the overall rank is $2 < n$.

That is to say that $\lambda = -3$ and $\lambda = -2$ cause the reachability pencil to lose rank.

Therefore, the system is not reachable/controllable (rank $\mathcal{R} = 1$) where $\lambda = -3$ and $\lambda = -2$ are the unreachable modes and α is the reachable mode.

Therefore, the system is stabilizable using feedback control because $\lambda = -3$ and $\lambda = -2$ are both in $\mathbb{C}^-$.

(c) We want to design a feedback controller that renders the closed-loop system asymptotically stable for all values of α. That is to say, we want to find a $u = Kx$ such that the closed-loop system is stable for all α.

$$u = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

However, since we know the system is unreachable with two unreachable modes, we can save computation.

If we partition A and B in this way, we have a sort of decomposition into a reachable and unreachable system (but B upside down)

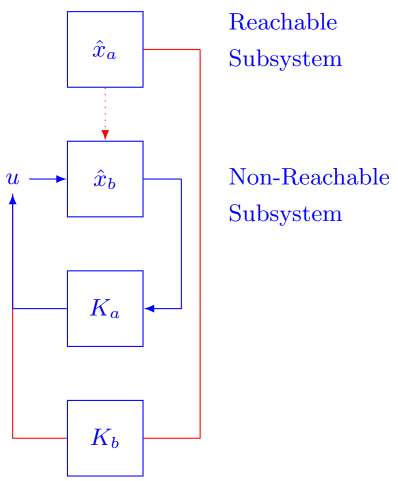
$$\dot{x} = \left[\begin{array}{cc|c} \overbrace{-3 \quad \alpha}^{x_a} & & 0 \\ 0 & -2 & 0 \\ \hline 0 & 0 & \underbrace{\alpha}_{x_b} \end{array}\right] x + \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right] u$$

This gives us the following equations

$$\dot{x}_a = A_a x_a$$

$$\dot{x}_b = A_b x_b + u$$

It is clear if you draw a block diagram that x_a does not communicate with the outside world without feedback.



And with feedback from x_a, it only connects x_a to x_b (red dotted line); therefore, u, as expected, is still not affecting x_a.

So we can set K_1 and K_2 equal to zero and pick K in the following way

$$u = \begin{bmatrix} 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_3 x_3 = \begin{bmatrix} 0 & 0 & k_3 \end{bmatrix} x$$

Therefore

$$A + BK = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & K_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \alpha + K_3 \end{bmatrix}$$

Now we want to choose K_3 such that $\alpha + K_3$ is negative.

A good selection:

$$K_3 = -\alpha - 1 \Rightarrow \alpha + K_3 = -1$$

Therefore

$$K = \begin{bmatrix} 0 & 0 & -\alpha - 1 \end{bmatrix}$$

Do note that if you include K_1 and K_2, then you could still save yourself a lot of computation if you spot when looking at

$$A + BK = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ K_1 & K_2 & \alpha + K_3 \end{bmatrix}$$

that K_1 and K_2 are not needed as they don't affect the eigenvalues.

The matrix is in lower triangular form, and therefore, K_1 and K_2 can take any value, i.e.

$$K = \begin{bmatrix} K_1 & K_2 & -\alpha - 1 \end{bmatrix}$$

Question 3. *Consider the following discrete-time system*

$$x[k+1] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} x[k]$$

$$y[k] = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} x[k]$$

(a) *Determine if the system is observable and find any unobservable and reachable modes.*

(b) *Assume the output of the system is $y[0] = y[1] = y[2] = 0$*

(i) *Compute all possible initial states, $x[0]$.*

(ii) *Compute all possible final states, $x[2]$.*

(iii) *Discuss the differences between part (i) and part (i) and what the results tell you about the system.*

(c) *Design and observer which places all eigenvalues at $\lambda = 0$ where for any $e[0]$, $e[k] = 0$ for $k \geq \bar{k}$. Determine the value of $\bar{k}$.*

Solution 3.

(a)

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Notice that the third row is the negative of the first ($R_1 = -R_3$). The middle row is linearly independent so the $\text{Rank}(\mathcal{O}) = 2$

Therefore, the observable subspace has rank 1, so we need to find the unobservable mode.

A simple way would be to spot that the eigenvalues are

$$\det \begin{bmatrix} \lambda & 1 & 0 \\ -1 & \lambda & -2 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda(\lambda^2 + 1) = 0$$

where

$$\lambda_1 = 0 \text{ and } \lambda_{1,2} = \pm j$$

Since there is only one unobservable mode and $\lambda_{1,2} = \pm j$ are complex conjugates of each other and share observability properties; the unobservable mode must be $\lambda_1 = 0$

Alternately, we could perform the PBH text, which would give us

$$\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \left[\begin{array}{ccc} s & 1 & 0 \\ -1 & s & -2 \\ 0 & 0 & s \\ \hline 1 & -1 & 2 \end{array}\right]$$

where $s = 0$ causes the observability pencil to lose rank as the third column is twice the first column ($C_1 = 2C_3$).

(b) We are told to assume

$$y[0] = y[1] = y[2] = 0$$

and we would like to compute all possible initial states. We know that

$$\begin{aligned}
y[0] &= Cx[0] + Dx[0] = Cx[0] = 0 \quad \text{(because } D = 0\text{)}\\
y[1] &= Cx[1] + Dx[0] = C(Ax[0] + Bu[0]) = CAx[0] = 0 \quad \text{(because } B = 0 \text{ and } D = 0\text{)}\\
y[2] &= Cx[2] = C(Ax[1]) = CA^2x[0] = 0
\end{aligned}$$

Notice that this is nothing more than

$$y = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} x[0] = \mathcal{O}x[0] = 0$$

Therefore, we can write

$$y = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & -2 \\ -1 & 1 & -2 \end{bmatrix} x[0] = 0$$

Solving the system of equations:

$$\begin{aligned}
x_1[0] - x_2[0] + 2x_3[0] &= 0\\
-x_1[0] - x_2[0] - 2x_3[0] &= 0
\end{aligned}$$

We have already shown that the last row is not linearly independent, so it does not give us any new information.

From the first equation, we get

$$x_1[0] = x_2[0] - 2x_3[0]$$

Substitute into the second equation:

$$\begin{aligned}
-(x_2[0] - 2x_3[0]) - x_2[0] - 2x_3[0] &= 0\\
\Rightarrow -x_2[0] + 2x_3[0] - x_2[0] - 2x_3[0] &= 0\\
\Rightarrow -2x_2[0] = 0 \Rightarrow x_2[0] &= 0
\end{aligned}$$

Substituting back into the first equation, we get

$$x_1[0] = -2x_3[0]$$

Therefore

$$x[0] = \begin{bmatrix} -2\alpha \\ 0 \\ \alpha \end{bmatrix}$$

(c) We are told to assume

$$y[0] = y[1] = y[2] = 0$$

and we would like to determine all the possible $x[2]$ states. We know that

$$\begin{aligned} x[1] &= Ax[0] \quad \text{(because } B = 0\text{)} \\ x[2] &= Ax[1] = A^2x[0] \end{aligned}$$

Recall that

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we need to calculate

$$A^2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we can calculate

$$x[2] = A^2x[0] = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2\alpha \\ 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 2\alpha - 2\alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) We are now asked to discuss the differences between the two results.

Notice that the possible initial states form a subspace, whereas the final state, $x[2]$, is a signal point at the origin. That is to say

$$x[0] = \begin{bmatrix} -2\alpha \\ 0 \\ \alpha \end{bmatrix}, \qquad x[2] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This highlights that the system is unobservable as the current state $x[0]$ can not be determined from future outputs. However, the system is reconstructable as we can reconstruct the state at $x[2]$ using current and past outputs.

(e) We now want to design an observer which places the eigenvalues at zero such that for any $e[0]$, $e(k) = 0$ for all $k \geq \bar{k}$. We would also like to determine $bark$.

Recall that

$$e[k] = x[k] - \hat{x}_e[k] = x[k] - \xi[k]$$

and

$$y_e[k] = C\xi[k] - Cx[k] = C\xi[k] - y[k]$$

where

$$\xi[k+1] = A\xi[k] + Ly_e = A\xi + LC\xi - Ly = (A + LC)\xi[k] - Ly$$

Therefore, we have to design L such that $A + LC$ has all its eigenvalues at zero.

We also know that the $e[k+1]$ is defined as follows

$$e[k+1] = Ax - A\xi - LC\xi + LCx = (A+LC)(x-\xi) = (A+LC)e[k]$$

Therefore if we want to find $\bar{k}$ we need to find the lowest value of $\bar{k}$ such that

$$(A+LC)^{\bar{k}} = 0$$

We start by letting

$$L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

Therefore

$$A + LC = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} L_1 & -L_1 & 2L_1 \\ L_2 & -L_2 & 2L_2 \\ L_3 & -L_3 & 2L_3 \end{bmatrix}$$

$$= \begin{bmatrix} L_1 & -1-L_1 & 2L_1 \\ 1+L_2 & -L_2 & 2+2L_2 \\ L_3 & -L_3 & 2L_3 \end{bmatrix}$$

However, again, notice that we can save computation by looking at

$$A = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 2 \\ \hline 0 & 0 & \underbrace{0}_{\lambda=0} \end{array}\right]$$

A is block upper triangular, so the eigenvalues are given by the eigenvalues of matrices

$$\begin{bmatrix} 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We have already shown that $\lambda = 0$ is an unobservable mode, so we only need to move the eigenvalues $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ using L_1 and L_2 so we can let

$$L = \begin{bmatrix} L_1 \\ L_2 \\ 0 \end{bmatrix} \quad \text{and} \quad A + LC = \begin{bmatrix} L_1 & -1-L_1 & 2L_1 \\ 1+L_2 & -L_2 & 2+2L_2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we want to find the characteristic polynomial of $A+LC$, which is given by

$$\det \begin{bmatrix} \lambda - L_1 & 1+L_1 & -2L_1 \\ -1-L_2 & \lambda + L_2 & -2-2L_2 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$= \lambda \left[(\lambda - L_1)(\lambda + L_2) + (L_2+1)(L_1+1) \right]$$

$$= \lambda \left[\lambda^2 - \lambda L_1 + \lambda L_2 - L_1 L_2 + L_1 + L_2 + 1\right]$$

$$= \lambda^3 + (L_2 - L_1)\lambda^2 + (L_1 + L_2 + 1)\lambda$$

In this case, we want the characteristic polynomial of $A + LC$ to be equal to λ^3 so that all the eigenvalues are at zero. Therefore

$$L_2 - L_1 = 0 \Rightarrow L_1 = L_2$$

and

$$L_1 + L_2 + 1 = 0 \Rightarrow L_1 = L_2 = -\frac{1}{2}$$

Thus

$$L = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Now we just need to determine $\bar{k}$. We now know that

$$A + LC = \begin{bmatrix} -1/2 & -1/2 & -1 \\ 1/2 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So lets first try $\bar{k} = 2$

$$(A + LC)^2 = \begin{bmatrix} -1/2 & -1/2 & -1 \\ 1/2 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/2 & -1/2 & -1 \\ 1/2 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore we have found $\bar{k}$ where

$$\bar{k} = 2$$