Lecture Exercises (part 1)

Question 1. *Consider the following electrical circuit, which is modelled as a linear, single-input, single-output, continuous-time system of dimension $n = 3$ using the state equations*

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{L_1}(u - x_3) && \text{(current through inductor } L_1\text{)} \\
\dot{x}_2 &= \frac{1}{L_2}(u - x_3) && \text{(current through inductor } L_2\text{)} \\
\dot{x}_3 &= \frac{1}{C}(x_1 + x_2) && \text{(voltage across the capacitor } C\text{)} \\
y &= x_1 + x_2
\end{aligned}$$

(a) Write the state-space representation of the system.

(b) Determine and justify whether the system is controllable.

(c) Determine and justify whether the system is observable.

(d) Find the unobservable subspace and determine physical property of the electrical circuit is non-distinguishable.

(e) Use the PBH test to find the unobservable modes of the system.

(f) Let $\hat{x} = T^{-1}x$ where

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the state space representation of the system with the state $\hat{x}$, identify the unobservable subsystem and explain why the system's eigenvalues coincide with the observable modes found in the previous part of the question.

Solution 1.

(a)

$$A = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} \\ 0 & 0 & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

(b)

$$R = \begin{bmatrix} \frac{1}{L_1} & 0 & -\frac{1}{L_1}(\frac{1}{CL_1} + \frac{1}{CL_2}) \\ \frac{1}{L_2} & 0 & -\frac{1}{L_2}(\frac{1}{CL_1} + \frac{1}{CL_2}) \\ 0 & \frac{1}{CL_1} + \frac{1}{CL_2} & 0 \end{bmatrix}$$

where

$$\det \mathcal{R} = 0$$

and the system is not controllable.

(c)

$$\mathcal{O} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -\frac{1}{L_1} - \frac{1}{L_2} \\ -\frac{1}{C}(\frac{1}{L_1} + \frac{1}{L_2}) & -\frac{1}{C}(\frac{1}{L_1} + \frac{1}{L_2}) & 0 \end{bmatrix}$$

where

$$\det \mathcal{O} = 0$$

and the system is not observable.

(d) Find all vectors $\bar{x}$ such that

$$\mathcal{O}\bar{x} = 0$$

where this would have one unique solution if the system were observable

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -\frac{1}{L_1} - \frac{1}{L_2} \\ \alpha & \alpha & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = 0$$

$$\begin{aligned} \bar{x}_1 + \bar{x}_2 &= 0 \\ (-\frac{1}{L_1} - \frac{1}{L_2})\bar{x}_3 &= 0 \implies \bar{x}_3 = 0 \\ \alpha(\bar{x}_1 + \bar{x}_2) &= 0 \implies \bar{x}_1 + \bar{x}_2 = 0 \end{aligned}$$

Therefore

$$\begin{bmatrix} \bar{x}_1 \\ -\bar{x}_1 \\ 0 \end{bmatrix}, \qquad \bar{x}_1 \neq 0$$

Therefore, the unobservable subspace is

$$\text{Im} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

It is also clear that the physical property that cannot be distinguished is the current through the two inductors in the circuit.

(e)

$$\text{rank}\, \mathcal{O} = 2$$

Therefore, there is only one unobservable mode, and it is an eigenvalue of A

$$\det(I\lambda - A) = \begin{bmatrix} \lambda & 0 & \frac{1}{L_1} \\ 0 & \lambda & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & \lambda \end{bmatrix}$$

which gives that charatrics polynomial

$$\lambda^3 + \frac{1}{L_1 C}\lambda + \frac{1}{L_2 C}\lambda = 0$$
$$\lambda\left(\lambda^2 + \frac{1}{L_1 C} + \frac{1}{L_2 C}\right) = 0$$

This gives us

$$\lambda = 0\,, \qquad \lambda = \pm j\sqrt{\frac{1}{L_1 C} + \frac{1}{L_2 C}}\,,$$

Where the unobservable mode has to be $\lambda = 0$ because complex conjugate pairs share the same properties.

Note that

$$\text{rank}\begin{bmatrix} I\lambda - A \\ C \end{bmatrix} = \text{rank}\begin{bmatrix} I\lambda - A \\ C \end{bmatrix}^* = \text{rank}\begin{bmatrix} I\lambda^* - A \\ C \end{bmatrix} \qquad \text{(because } A \text{ and } C \text{ are real)}$$

So if the matrix pencil loses rank for λ then it will lose λ^*

(f) We are given that

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

We also know that

$$\dot{x} = Ax + Bu\,, y = Cx$$

Therefore

$$\dot{\hat{x}} = T^{-1}x = T^{-1}(Ax + Bu) = T^{-1}(AT\hat{x} + Bu)\,, \qquad y = CT\hat{x}$$

which gives us

$$\hat{A} = T^{-1}AT\,, \qquad \hat{B} = T^{-1}B\,, \qquad \hat{C} = CT$$

So now we need to find the inverse of the matrix T

$$T^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

We can now calculate the values of $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ which are

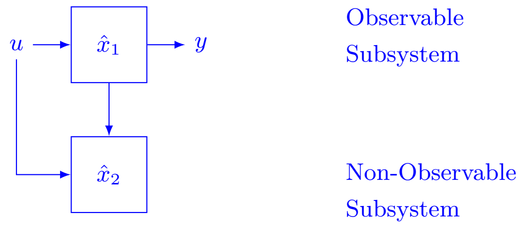
$$\hat{A} = \left[\begin{array}{cc|c} 0 & \frac{1}{c} & 0 \\ -\frac{1}{L_1} - \frac{1}{L_2} & 0 & 0 \\ \hline -\frac{1}{L_1} & 0 & 0 \end{array}\right], \qquad \hat{B} = \left[\begin{array}{c} 0 \\ 2 \\ \hline 1 \end{array}\right], \qquad \hat{C} = \left[\begin{array}{cc|c} 0 & 1 & 0 \end{array}\right]$$

So if I partition $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$, where $\hat{x}_1$ has two components and $\hat{x}_2$ has one component.

Then we have a system described by

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{A}_{11}\hat{x}_1 + \hat{B}_1 u \\ \dot{\hat{x}}_2 &= \hat{A}_{12}\hat{x}_1 + \underbrace{\hat{A}_{22}\hat{x}_2}_{=0} + u \\ y &= \hat{C}_1\hat{x}_1\end{aligned}$$

where $\hat{A}_{22}$ is the unobservable mode of the system



Question 2. *Consider the discrete-time system*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \qquad D = -1$$

(a) Determine and justify whether the system is reachable.

(b) Determine and justify whether the system is observable.

(c) Assume $u[0] = u[1] = u[2] = 1$ which give the output signal $y[0] = -5,\ y[1] = 0,\ y[2] = 1$. Find the free response of the output for $k = \{0, 1, 2\}$.

(d) Find a sequence that drives the initial state $x[0]$ to zero and what is the smallest number of steps needed.

Solution 2.

(a)

$$\mathcal{R} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix}$$

which clearly has rank 3. Therefore, the system is reachable.

(b)

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

which clearly has rank 3. Therefore, the system is observable.

(c) Recall the free response is for $u[k] = 0$ for all k. However, we need to know the initial state of the system.

This is because the free output response is

$$y[0] = Cx[0] + \underbrace{Du[0]}_{=0}$$
$$y[1] = Cx[1] = C(Ax[1] + \underbrace{Bu[1]}_{=0}) = CAx[0]$$
$$y[2] = CA^2x[0]$$

So, to find the initial state, first we know that $y[0] = Cx[0] + Du[0]$ therefore

$$-5 = x_1[0] - x_3[0] - \underbrace{u[0]}_{=1} = x_1[0] - x_3[0] - 1$$

Second we know that, $y[1] = Cx[1] + Du[1] = C(Ax[0] + Bu[0]) + Du[1]$ therefore first compute

$$CA = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$$

Then we get

$$0 = x_2[0] - x_3[0] + 2\underbrace{u[0]}_{=1} - \underbrace{u[1]}_{=1} = x_2[0] - x_3[0] + 1$$

Third we know that $y[2] = Cx[2] + Du[2] = CA^2x[0] + CABu[0] + CBu[1] + Du[2]$ therefore we need to compute

$$CA^2 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

and

$$CAB = 3$$

Then we get

$$1 = -x_3[0] + 3\underbrace{u[0]}_{=1} - 2\underbrace{u[1]}_{=1} - \underbrace{u[2]}_{=1} = -x_3[0] + 4 \implies x_3 = 3$$

So, we now have three equations and three unknowns:

$$1 = -x_3[0] + 4 \implies x_3 = 3$$

$$0 = x_2[0] - x_3[0] + 1 \implies x_2[0] = x_3[0] - 1 = 2$$
$$-5 = x_1[0] - x_3[0] - 1 \implies x_1[0] = x_3[0] - 4 = -1$$

Therefore

$$x[0] = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, recall that the free output response is

$$\begin{aligned} y[0] = Cx[0] &= \begin{bmatrix}1 & 0 & -1\end{bmatrix} \begin{bmatrix}-1\\2\\3\end{bmatrix} = -4 \\ y[1] = CAx[0] &= \begin{bmatrix}0 & 1 & -1\end{bmatrix} \begin{bmatrix}-1\\2\\3\end{bmatrix} = -1 \\ y[1] = CA^2x[0] &= \begin{bmatrix}0 & 0 & -1\end{bmatrix} \begin{bmatrix}-1\\2\\3\end{bmatrix} = -3 \end{aligned}$$

(d) We want to now find a sequence that drives the initial state $x[0]$ to zero and what is the smallest value of k needed.

We know that the maximum number of steps needed is n (i.e. the size of the state space because the system is controllable).

We need to find out if we can drive the state to zero in fewer than three steps,

So, let's start with just one step where $k = 1$

$$x[1] = Ax[0] + Bu[0]$$

which gives us

$$x[1] = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}-1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\1\\-2\end{bmatrix} u[0] = \begin{bmatrix}2\\0\\3\end{bmatrix} + \begin{bmatrix}0\\1\\-2\end{bmatrix} u[0] = \begin{bmatrix}2\\u[0]\\3-2u[0]\end{bmatrix}$$

We cannot make $x[1] = 0$ with any $u[0]$

What about two steps where $k = 2$

$$x[2] = Ax[1] + Bu[1]$$

which gives us

$$x[2] = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}2\\u[0]\\3-2u[0]\end{bmatrix} + \begin{bmatrix}0\\1\\-2\end{bmatrix} u[1] = \begin{bmatrix}u[0]\\u[1]\\3-2u[0]-2u[1]\end{bmatrix}$$

Again we cannot make $x[2] = 0$ with any selection of $u[1]$ and $u[2]$

Therefore, we now know that the minimum number of steps is equal to n (i.e. three steps)

To find the sequence that drives the initial state $x[0] = 0$ we now need to workout when $k = 3$

$$x[3] = Ax[2] + Bu[2]$$

which gives us

$$x[3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ 3 - 2u[0] - 2u[1] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} u[2] = \begin{bmatrix} u[1] \\ u[2] \\ 3 - 2u[0] - 2u[1] - 2u[2] \end{bmatrix}$$

Now we can just find the values of $u[0]$, $u[1]$, $u[2]$ which solve the equations

$$u[1] = 0$$
$$u[2] = 0$$
$$3 - 2u[0] - 2u[1] - 2u[2] = 0 \implies 3 - 2u[0] = 0 \implies u[0] = \frac{3}{2}$$

Therefore, we have found the sequence that drives the initial state to zero.

$$u[k] = \{\frac{3}{2}\ ,\ 0\ ,\ 0\}$$

Question 3. *Consider the continuous-time system*

$$A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

(a) Determine and justify whether the system is controllable as a function of α.

(b) Design an output feedback controller applying state feedback of gain K such that the matrix $A + BK$ has all its eigenvalues equal to $\lambda = -1$.

(c) Determine and justify whether the system is observable as a function of α.

(d) Design an asymptotic observer for the system. Select the output injection gain L such that the matrix $A + LC$ has all its eigenvalues equal to $\lambda = -2$.

Solution 3.

(a)

$$\mathcal{R} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

Therefore $\det \mathcal{R} = 1$, and the system is controllable for all α.

(b)

$$A + BK = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} \alpha + K_1 & K_2 \\ 1 & 1 \end{bmatrix}$$

$$\det(\lambda I - (A + BK)) = \det \begin{bmatrix} \lambda - \alpha - K_1 & -K_2 \\ -1 & \lambda - 1 \end{bmatrix} = 0$$

$$(\lambda - \alpha - K_1)(\lambda - 1) - K_2 = 0$$

$$\lambda^2 - \alpha\lambda - K_1\lambda - \lambda + \alpha + K_1 - K_2 = 0$$

$$\lambda^2 + (-\alpha - K_1 - 1)\lambda + \alpha + K_1 - K_2 = 0$$

This needs to equal the following so that both eigenvalues are at -1

$$(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1$$

Therefore

$$2 = -\alpha - K_1 - 1 \implies K_1 = -\alpha - 3$$

and

$$1 = \alpha + K_1 - K_2 \implies K_2 = \alpha + (-\alpha - 3) - 1 = -4$$

(c)

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

where the rank $\mathcal{O} = 2$ for all α, and the system is observable for all α.

(d) An observer is given by

$$\dot{\xi} = (A + LC)\xi - Ly + Bu$$

where

$$x_e = x = \xi$$

First we have

$$A + LC = \begin{bmatrix} \alpha & L_1 \\ 1 & 1 + L_2 \end{bmatrix}$$

Then we find

$$\det(\lambda I - (A + LC)) = \det \begin{bmatrix} \lambda - \alpha & -L_1 \\ -1 & \lambda - 1 - L_2 \end{bmatrix} = 0$$

which gives us

$$(\lambda - \alpha)(\lambda - 1 - L_2) - L_1 = 0$$

$$\lambda^2 - \lambda - \lambda L_2 - \alpha\lambda + \alpha + \alpha L_2 - L_1 = 0$$

$$\lambda^2 + (-1 - L_2 - \alpha)\lambda + \alpha + \alpha L_2 - L_1 = 0$$

This needs to equal the following so that both eigenvalues are at -2

$$(\lambda + 2)^2 = \lambda^2 + 4\lambda + 4$$

Therefore

$$4 = -1 - L_2 - \alpha \implies L_2 = -5 - \alpha$$

and

$$4 = \alpha + \alpha L_2 - L_1 \implies L_1 = \alpha + \alpha(-5 - \alpha) - 4 = \alpha(-4 - \alpha) - 4$$

Recall that we have

$$\dot{x} = Ax + Bu$$

We define

$$e = x - x_e$$

where we want

$$\dot{e} = (A - LC)e$$

and we want $e(t)$ to converge to zero for all $x(0)$ $\xi(0)$ and $u(t)$ (which is sometimes called uniform convergence).

Note that we want the observer to be faster than the state feedback, but we also need to consider the 'peaking phenomenon'. This 'peaking phenomenon' occurs because the energy of the error is constant.