SOLUTIONS AND MARKING SCHEME

Queen Mary
University of London

May Examination Period 2025

ECS778P Main ***ADVANCED CONTROL SYSTEMS*** Duration: 2 hours (+1 for uploads)

There are FOUR questions on this paper

Answer ALL questions

Each question carries 25% of the marks.

This paper requires **two hours work**. There is an extra hour allowance for downloading the paper and uploading your answers.

You MUST submit your answers before the exam end time.

You must follow the online exam guidelines and instructions on the EECS exam access and submission page.

This is an open-book exam. You may use lecture notes and any module materials made available to you (online or physical). You must not use other online resources.

YOU MUST COMPLETE THE EXAM ON YOUR OWN, WITHOUT CONSULTING OTHERS.

Examiners:
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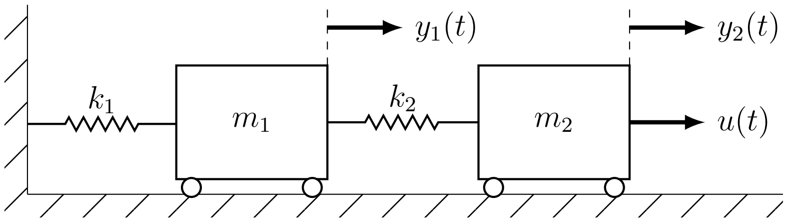
Question 1 **[25 marks]**

(a) Consider the translational mechanical system shown below, where $y_1(t)$ and $y_2(t)$ denote the displacements of the associated masses from their static equilibrium positions, and $u(t)$ represents a force applied to the second mass m_2.



The system's parameters are the masses m_1 and m_2 and the spring stiffnesses k_1 and k_2. The input is the applied force $u(t)$, and the outputs are taken as the mass displacements. Given that the state variables are defined as

$$x_1(t) = y_1(t), \quad x_2(t) = y_2(t) - y_1(t), \quad x_3(t) = \dot{y}_1(t), \quad x_4(t) = \dot{y}_2(t).$$

Derive the state-space realization for the mechanical system. That is, derive the coefficient matrices A, B, C, and D.

[9 marks]

Solution:

Newton's second law applied to each mass yields:

$$m_1\ddot{y}_1(t) = k_2[y_2(t) - y_1(t)] - k_1 y_1(t), \; \boxed{2}$$

$$m_2\ddot{y}_2(t) = u(t) - k_2[y_2(t) - y_1(t)]. \; \boxed{2}$$

In terms of state-space variables:

$$\dot{x}_3 = \frac{k_2}{m_1}x_2 - \frac{k_1}{m_1}x_1 \; \boxed{1}$$

$$\dot{x}_4 = \frac{1}{m_2}u - \frac{k_2}{m_2}x_2 \; \boxed{1}$$

The state-space representation is:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -\frac{k_1}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ 0 & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u(t), \; \boxed{2}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t). \boxed{1}$$

Difficulty: Basic

Marking criteria: As above

(b) Is the following statement True or False:

An observer is a special type of filter designed to estimate the state of a system based on the systems inputs and outputs.

[4 marks]

Solution: True

Difficulty: Basic

Marking criteria: As above

(c) Is the following statement True or False:

It is possible to make an uncontrollable system controllable using feedback.

[4 marks]

Solution: False

Difficulty: Basic

Marking criteria: As above

(d) Is the following statement True or False:

For a continuous time system, if all the eigenvalues λ with a positive real part are reachable modes of the system, then the system is stabilizable.

[4 marks]

Solution: True

Difficulty: Basic

Marking criteria: As above

(e) Is the following statement True or False:

It is possible for a linear system to have two distinct equilibrium points.

[4 marks]

Solution: False

Difficulty: Basic

Marking criteria: As above

Question 2 **[25 marks]**

Consider the following electrical circuit, which is modelled as a linear, single-input, single-output, continuous-time system of dimension $n = 2$ using the state equations

$$\dot{x}(t) = Ax(t) + Bu(t) \,, \qquad y(t) = Cx(t) + Du(t) \,,$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C_1} \\ -\frac{1}{L_1} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

and C_1 and L_1 are positive constants representing the circuit's capacitance and inductance, respectively.

(a) Determine the reachability and observability properties of the electrical circuit.

[6 marks]

Solution:

Reachability is determined by the reachability matrix, $R = \begin{bmatrix} B & AB \end{bmatrix}$, which is given by:

$$R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{L_1} \end{bmatrix}. \; \boxed{1}$$

To check for reachability, we need to determine if $\det(R) \neq 0$ (i.e. rank$(R) = 2$) where $\det(R) = -\frac{1}{L_1}$. $\boxed{1}$

L_1 is a positive constant, therefore, **the system is reachable**. $\boxed{1}$

Observability is determined by the observability matrix, $O = \begin{bmatrix} C \\ CA \end{bmatrix}$, which is given by:

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{C_1} \end{bmatrix}. \; \boxed{1}$$

To check for observability, we need to determine if $\det(O) \neq 0$ (i.e. rank$(O) = 2$) where $\det(O) = \frac{1}{C_1}$. $\boxed{1}$

C_1 is a positive constant, therefore, **the system is observable** $\boxed{1}$

Difficulty: Intermediate

Marking criteria: As above

(b) Express the state-space representation of the electrical circuit in controllable canonical form.

[4 marks]

Solution:

The controllable canonical form of a 2×2 matrix is given by

$$A_c = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

where α_0 and α_1 are the coefficients of the characteristic polynomial,

$$\lambda^2 + \alpha_1 \lambda + \alpha_0 = \lambda^2 + \frac{1}{C_1 L_1}. \quad \boxed{2}$$

In this case $\alpha_1 = 0$ and $\alpha_0 = \frac{1}{C_1 L_1}$. Therefore,

$$A_c = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C_1 L_1} & 0 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad \boxed{2}$$

Difficulty: Basic

Marking criteria: As above

(c) Let $u(t) = 0$, for all $t \geq 0$. Compute the equilibrium points of the electrical circuit.

[4 marks]

Solution:

To begin with, note $\det(A) = \frac{1}{C_1 L_1}$. Since C_1 and L_1 are both positive constants the matrix A is always invertible. $\boxed{2}$

Hence the only equilibrium for $u(t) = u(0) = 0$ is $x(0) = -A^{-1}Bu(0) = 0$ for all $t \geq 0$. $\boxed{2}$

Difficulty: Basic

Marking criteria: As above

(d) Assume now $u(t) = u(0)$, for all $t \geq 0$, where $u(t) \neq 0$. Compute the equilibrium points of the electrical circuit.

[6 marks]

Solution:

As for part (c), the matrix A is always invertible hence the only equilibrium for $u(t) = u(0) \neq 0$ is $x(0) = -A^{-1}Bu(0) = 0$ for all $t \geq 0$. $\boxed{2}$

Note that $A^{-1} = -C_1 L_1 \begin{bmatrix} 0 & -\frac{1}{C_1} \\ \frac{1}{L_1} & 0 \end{bmatrix}$ **2** , therefore,

$$x(0) = -C_1 L_1 \begin{bmatrix} 0 & -\frac{1}{C_1} \\ \frac{1}{L_1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0) = -C_1 L_1 \begin{bmatrix} 0 \\ \frac{1}{L_1} \end{bmatrix} u(0) = \begin{bmatrix} 0 \\ -C_1 u(0) \end{bmatrix} \boxed{2}$$

Difficulty: Intermediate

Marking criteria: As above

(e) Determine and justify the Lyapunov stability of the electrical circuit.

[5 marks]

Solution:

The characteristic polynomial is given by

$$\det(\lambda I - A) \det\left(\begin{bmatrix} \lambda & -\frac{1}{C_1} \\ \frac{1}{L_1} & \lambda \end{bmatrix}\right) = 0 \boxed{1}$$

$$\lambda^2 + \frac{1}{C_1 L_1} = 0$$
$$\lambda = \pm i \sqrt{\frac{1}{C_1 L_1}} \boxed{1}$$

The two roots are on the imaginary axis, so the system/equilibrium is either stable *(marginally stable)* or unstable. **1**

To check, we can express the roots characteristic polynomial in the following way

$$p(\lambda) = \left(\lambda - i\frac{1}{\sqrt{C_1 L_1}}\right)\left(\lambda + i\frac{1}{\sqrt{C_1 L_1}}\right).$$

It is clear from this expression that the algebraic and geometric multiplicity of both eigenvalues is equal to 1. **2**

Therefore, the system/equilibrium is stable (but not asymptotically stable).

Difficulty: Intermediate

Marking criteria: As above

Question 3 **[25 marks]**

Consider a linear, single-input, single-output, discrete-time system of dimension $n = 3$, described by the following state equations

$$x[k+1] = Ax[k] + Bu[k] \ , \qquad y[k] = Cx[k] + Du[k] \ ,$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \quad C = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} , \quad D = 1 \ ,$$

and let x_0 be the initial state of the system, that is $x_0 = x[0]$.

(a) Determine and justify whether the system is reachable, controllable, and observable.

[6 marks]

Solution:

The reachability matrix is

$$R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} . \ \boxed{1}$$

The rank of R is $2 < n$. Hence, the system is not reachable. $\boxed{1}$

Note now that

$$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \ \boxed{1}$$

hence $\text{Im}(A^3) \not\subset \text{Im}(R)$, which implies that the system is non-controllable. $\boxed{1}$

Alternative Method

The reachability pencil is

$$\left[\ sI - A \,\middle|\, B \ \right] = \left[\begin{array}{ccc|c} s & 0 & -1 & 0 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s-1 & 1 \end{array}\right] . \ \boxed{2}$$

The reachability pencil loses rank at $s = 1$, indicating the presence of an unreachable mode at $s = 1$. As a result, the system is **non-reachable**. $\boxed{1}$

Furthermore, since this unreachable mode is not at zero, the system is also **non-controllable**. $\boxed{1}$

The observability matrix

$$O = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \boxed{\mathbf{1}}$$

The rank of O is $1 < n$. Hence, the system is not observable. $\boxed{\mathbf{1}}$

Difficulty: Intermediate

Marking criteria: As above

(b) Using the PBH test, determine the unreachable modes of the system.

[3 marks]

Solution:

As A is upper triangular the eigenvalues of A can just be read directly off the diagonal where $\lambda_1 = 0$ and $\lambda_{2,3} = 1$. $\boxed{\mathbf{1}}$

The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s & 0 & -1 & 0 \\ 0 & s-1 & & 0 \\ 0 & 0 & s-1 & 1 \end{array}\right] . \boxed{\mathbf{1}}$$

To find the unreachable modes, it must be determined for which eigenvalues the reachability pencil loses rank.

The reachability pencil loses rank when $s = 1$. Hence, there is an unreachable mode at $s = 1$. $\boxed{\mathbf{1}}$

Difficulty: Advanced

Marking criteria: As above

(c) Assume $u[0] = 0$. Determine all initial conditions $x[0]$ such that $x[1] = 0$.

[3 marks]

Solution:

Note that, since $u[0] = 0$,

$$x[1] = Ax[0] \boxed{\mathbf{1}} = \begin{bmatrix} x_3[0] \\ x_2[0] \\ x_3[0] \end{bmatrix} = 0 \ , \boxed{\mathbf{1}}$$

hence all initial conditions described by $x[0] = \begin{bmatrix} * & 0 & 0 \end{bmatrix}^T$ with * any number, are such that $x[1] = 0$. **1**

Difficulty: Basic

Marking criteria: As above

(d) Assume $u[0] = u[1] = 0$. Determine all initial conditions $x[0]$ such that $x[2] = 0$.

[3 marks]

Solution:

Note that, since $u[0] = u[1] = 0$,

$$x[2] = A^2 x[0] \; \boxed{1} = \begin{bmatrix} x_3[0] \\ x_2[0] \\ x_3[0] \end{bmatrix}, \; \boxed{1}$$

hence all initial conditions described by $x[0] = \begin{bmatrix} * & 0 & 0 \end{bmatrix}^T$ with * any number, are such that $x[2] = 0$. **1**

Difficulty: Basic

Marking criteria: As above

(e) Let $x[0] = 0$. Assume that $y[0] = 0$, $y[1] = 0$ and $y[2] = 1$.

(i) Determine input values $u[0]$, $u[1]$, and $u[2]$ which generate this output sequence.

[6 marks]

Solution:

Note that

$$y[0] = Cx[0] + Du[0] \implies y[0] = u[0] \quad \text{(since } x[0] = 0 \text{ and } D = 1) \; \boxed{1}$$

$$y[1] = Cx[1] + Du[1] \quad \text{where} \quad x[1] = Bu[0] \quad \text{(since } x[0] = 0)$$
$$y[1] = CBu[0] + u[1] \quad \text{(since } D = 1)$$
$$y[1] = u[0] + u[1] \; \boxed{1}$$

$$y[2] = Cx[2] + Du[2]$$
$$\text{where} \quad x[2] = ABu[0] + Bu[1] \quad \text{(since } x[1] = Bu[0])$$
$$y[2] = CABu[0] + CBu[1] + u[2] \quad \text{(since } D = 1)$$
$$y[2] = u[0] + u[1] + u[2] \; \boxed{1}$$

Therefore

$$y[0] = u[0], \quad y[1] = u[0] + u[1], \quad y[2] = u[0] + u[1] + u[2].$$

Hence, the conditions $y[0] = y[1] = 0$ and $y[2] = 1$ yield $u[0] = 0$ **1**, $u[1] = 0$ **1** and $u[2] = 1$ **1**.

Difficulty: Intermediate

Marking criteria: As above

(ii) Suppose that the input sequence determined in part (i) is completed with the values $u[k] = 0$, for all $k \geq 3$. Determine the output sequence $y[k]$ for all $k \geq 3$.

[2 marks]

Solution:

The output sequence resulting from the above input sequence, extended with $u[k] = 0$, for all $k \geq 3$, is

$$y[3] = 1, \quad y[4] = 1, \quad \ldots$$

Difficulty: Intermediate

Marking criteria: As above or full marks for any mention of the correct output sequence

(iii) Explain why the output sequence is constant for all $k \geq 3$.

[2 marks]

Solution:

The reason why the output sequence is constant is that the state $x[3] = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ is an *equilibrium* of the system for $u[k] = 0$.

Hence, the state of the system is driven from $x[0] = 0$ to $x[3] = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ by the input $u[0] = 0$, $u[1] = 0$ and $u[2] = 1$ and remains there for all $t \geq 3$.

Difficulty: Advanced

Marking criteria: As above or full marks for any mention that $x[3] = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ is an *equilibrium* of the system for $u[k] = 0$

Question 4 **[25 marks]**

Consider a linear, single-input, single-output, continuous-time system of dimension $n = 3$ described by the following state equations

$$\dot{x}_1(t) = x_2(t) \ ,$$
$$\dot{x}_2(t) = x_3(t) \ ,$$
$$\dot{x}_3(t) = -\alpha x_3(t) + u(t) \ ,$$

with the output equation

$$y(t) = x_1(t) \ .$$

(a) Show, using the PBH test, that the system is observable for all α.

[5 marks]

Solution:

Note that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix}, \ \boxed{\mathbf{1}} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \boxed{\mathbf{1}}$$

The observability pencil is

$$\left[\begin{array}{ccc} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s+\alpha \\ \hline 1 & 0 & 0 \end{array}\right], \ \boxed{\mathbf{2}}$$

which has rank 3 for any s and any α.

Hence, the system is always observable. $\boxed{\mathbf{1}}$

Difficulty: Advanced

Marking criteria: As above

(b) Design an asymptotic observer for the system. Select the output injection gain L such that the matrix $A + LC$ has three eigenvalues equal to $\lambda = -2$.

[10 marks]

Solution:

Note that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} , \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} . \boxed{\mathbf{1}}$$

An asymptotic observer is described by

$$\dot{\xi} = A\xi + L(C\xi - y) = (A + LC)\xi - Ly \ , \boxed{\mathbf{1}}$$

for some $L = [L_1 \ L_2]'$, where ξ is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. Note that

$$A + LC = \begin{bmatrix} L_1 & 1 & 0 \\ L_2 & 0 & 1 \\ L_3 & 0 & -\alpha \end{bmatrix} , \boxed{\mathbf{1}}$$

and its characteristic polynomial is given by

$$\det \left(\begin{bmatrix} \lambda - L_1 & -1 & 0 \\ -L_2 & \lambda & -1 \\ -L_3 & 0 & \lambda + \alpha \end{bmatrix} \right) = 0 \ , \boxed{\mathbf{1}}$$

where

$$\lambda^3 + (\alpha - L_1)\lambda^2 - (L_2 + L_1\alpha)\lambda - (L_3 + L_2\alpha) = 0 \boxed{\mathbf{2}}$$

This should be equal to $(\lambda + 2)^3 = \lambda^3 + 6\lambda^2 + 12\lambda + 8$ $\boxed{\mathbf{1}}$, yielding

$L_1 = \alpha - 6$, $\boxed{\mathbf{1}}$ $\quad L_2 = -12 + (6 - \alpha)\alpha$, $\boxed{\mathbf{1}}$ $\quad L_3 = -8 + 12\alpha - (6 - \alpha)\alpha^2$. $\boxed{\mathbf{1}}$

Difficulty: Advanced

Marking criteria: As above

(c) Design an output feedback controller applying state feedback of gain K such that the matrix $A + BK$ has three eigenvalues equal to $\lambda = -1$.

[10 marks]

Solution: Note that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} , \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \boxed{\mathbf{1}}$$

Let $K = [K_1 \; K_2 \; K_3]$ and note that

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ K_1 & K_2 & K_3 - \alpha \end{bmatrix}, \boxed{1}$$

and that the characteristic polynomial is given by,

$$\det(I\lambda - (A + BK)) = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -K_1 & -K_2 & \lambda - K_3 + \alpha \end{bmatrix} = 0 \; , \boxed{2}$$

and therefore, $\lambda^3 + (\alpha - K_3)\lambda^2 - K_2\lambda - K_1 = 0$. $\boxed{2}$

This should be equal to $(\lambda+1)^3 = \lambda^3+3\lambda^2+3\lambda+1$ $\boxed{1}$, such that all eigenvalues of $A + BK$ are equal to -1, yielding, the selection is

$$K_1 = -1 \; , \boxed{1} \qquad K_2 = -3 \; , \boxed{1} \qquad K_3 = \alpha - 3 \; . \boxed{1}$$

Difficulty: Advanced

Marking criteria: As above

End of questions