Week 9 Tutorial

Question 1. *Consider the discrete-time system $x(k+1) = Ax(k)$, $y(k) = Cx(k)$. Let*

$$A = \begin{bmatrix} 0 & -4 & 0 \\ 1 & 4 & 0 \\ 0 & -4 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Determine if the system is observable and compute the unobservable subspace.

Solution 1. The observability matrix is

$$O = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 0 \\ 4 & 12 & 0 \end{bmatrix}.$$

This matrix has rank two, hence the system is not observable. The unobservable subspace $\ker O$ is spanned by the vector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

this means that it is not possible to obtain information on the third component of the state from measurements of the output.

Question 2. *Consider the continuous-time system*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 3 & -1+\epsilon \\ 1 & 2-\epsilon \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 1 \end{bmatrix} x. \end{aligned}$$

(a) Show that the system is observable for all $\epsilon \neq 1/2$.

(b) Let $\epsilon = 1/2$. Determine, using PBH test, the unobservable modes.

Solution 2.

(a) The observability matrix is

$$O = \begin{bmatrix} -1 & 1 \\ -2 & 3-2\epsilon \end{bmatrix}.$$

Note that $\det(O) = 2\epsilon - 1$. Therefore the system is observable if $\epsilon \neq 1/2$.

(b) The observability pencil, for $\epsilon = 1/2$, is

$$\left[\begin{array}{cc} s-3 & 1/2 \\ -1 & s-3/2 \\ \hline -1 & 1 \end{array}\right].$$

As the system is not observable we know that the observability pencil loses rank, i.e. it has rank equal to one, for some s. To compute such an s consider the submatrix

$$\left[\begin{array}{cc} -1 & s-3/2 \\ \hline 1 & -1 \end{array}\right].$$

This has rank equal to one for $s = 5/2$, which is therefore the unobservable mode.

Question 3. *Consider the linear electrical network in Figure A. Let u be the driving voltage.*

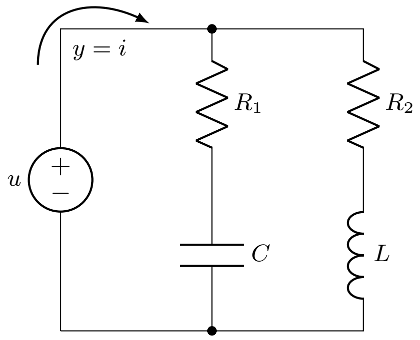

Figure A: The electrical network for Question 3.

(a) *Using Kirchhoff's laws, or otherwise, express the dynamics of the circuit in the standard state-space form*

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx + Du$$

Take x_1 to be the voltage across the capacitor, x_2 to be the current through the inductor and the output to be the current supplied by the generator.

(b) *Derive a condition on the parameters R_1, R_2, C and L under which the pair (A, C) is observable.*

(c) *Assume $R_1R_2C = L$. Define the unobservable subspace. Illustrate this subspace as lines in $\mathcal{R}^2$.*

Solution 3. Let x_1 denote the voltage across C and x_2 the current through L.

(a) Kirchhoff's laws yield

$$u = x_1 + R_1C_1\dot{x}_1 \qquad\qquad u = R_2x_2 + l\dot{x}_2$$

and

$$y = i = x_2 + \frac{u - x_1}{R_1}.$$

As a result,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu = \begin{bmatrix} -\frac{1}{R_1C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C} \\ \frac{1}{L} \end{bmatrix} u$$

and

$$y = Cx + Du = \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix} x + \frac{1}{R_1}u.$$

(b) The observability matrix is

$$O = \begin{bmatrix} -\frac{1}{R_1} & 1 \\ \frac{1}{R_1^2 C} & -\frac{R_2}{L} \end{bmatrix}$$

and

$$\det(O) = \frac{1}{R_1}\left(\frac{R_2}{L} - \frac{1}{R_1 C}\right).$$

Hence, the system is observable if, and only if,

$$R_1 R_2 C \neq L.$$

(c) When $R_1 R_2 C = L$ the observability matrix is

$$O = \begin{bmatrix} -\frac{1}{R_1} & 1 \\ \frac{1}{R_1^2 C} & -\frac{1}{R_1 C} \end{bmatrix}$$

To find the unobservable subspace

$$\begin{bmatrix} -\frac{1}{R_1} & 1 \\ \frac{1}{R_1^2 C} & -\frac{1}{R_1 C} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, looking at the top row, the unobservable subspace is $v_1 = R_1 v_2$, which is indicated in the figure below.