Week 8 Tutorial

Question 1. *Consider the discrete-time system $x[k+1] = Ax[k] + Bu[k]$ from last week. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Using PHB test determine the unreachable modes.

(b) Show that the system is controllable.

Solution 1.

(a) The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s & -1 & 0 & 1 \\ 1 & s & 0 & -1 \\ 0 & 0 & s & 0 \end{array}\right].$$

This matrix has rank three for all $s \neq 0$, hence, the unreachable mode is $s = 0$.

(b) The system is controllable since the unreachable modes are at $s = 0$. Note that

$$A^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$\text{Im}A^3 \subset \text{Im}R,$$

which also proves the system is controllable.

Question 2. *Consider the continuous-time system $x[k+1] = Ax[k] + Bu[k]$ from last week. Let*

$$A = \begin{bmatrix} 0 & \gamma \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(a) Show using PHB test that the system is reachable.

(b) Express the state-space representation of the electrical system in controllable canonical form.

Solution 2.

(a) The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{cc|c} s & -\gamma & 1 \\ 1 & s & 0 \end{array}\right].$$

which has rank two for any s and any γ.

Hence, the system is always **reachable**.

(b) The controllable canonical form of a 2×2 matrix is given by

$$A_c = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

where α_0 and α_1 are the coefficients of the characteristic polynomial,

$$\lambda^2 + \alpha_1\lambda + \alpha_0 = \det(I\lambda - A) = \det\begin{bmatrix} \lambda & -\gamma \\ 1 & \lambda \end{bmatrix} = \lambda^2 + \gamma = 0$$

In this case $\alpha_1 = 0$ and $\alpha_0 = \gamma$. Therefore,

$$A_c = \begin{bmatrix} 0 & 1 \\ -\gamma & 0 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Question 3. *Consider the discrete-time system* $x[k+1] = Ax[k] + Bu[k]$. *Let*

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Using PHB test determine the unreachable modes.

(b) Comment on whether the system is controllable.

Solution 3.

(a) The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s+3 & -1 & 0 & 1 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s+2 & 0 \end{array}\right].$$

This matrix loses rank for $s = 1$ and $s = -2$, hence, these are unreachable modes of the system

(b) The system is not controllable since the unreachable modes are not at $s = 0$. Also note that

$$A^3 = \begin{bmatrix} -27 & 7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix},$$

hence

$$\text{Im}A^3 \not\subset \text{Im}R,$$

which proves the system is not controllable.