Week 6 Tutorial

Question 1. *Consider the discrete-time system* $x[k+1] = Ax[k] + Bu[k]$*. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(a) *Compute the reachability matrix* R*.*

(b) *Determine if the system is reachable and compute the set of reachable states.*

(c) *Determine all states* x_I *such that* $x[0] = x_I$ *and* $x[1] = 0$*.*

Solution 1.

(a) The reachability matrix is

$$R = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 2 & -2 & -2 \end{bmatrix}.$$

(b) The first two columns of the reachability matrix are linearly independent and $\det(R) = 0$, hence, the system is not reachable. The set of reachable states is two-dimensional, and it is described by the linear combination of the first two columns of the reachable matrix.

(c) We have to determine all states which are controllable in one step. Instead of using the definition of controllable states in one step, let's perform a direct calculation. Let

$$x[0] = x_I = \begin{bmatrix} x_{I,1} \\ x_{I,2} \\ x_{I,3} \end{bmatrix}$$

and note that

$$x[1] = Ax[0] + Bu[0] = \begin{bmatrix} x_{I,2} + u[0] \\ -x_{I,1} - u[0] \\ 2(x_{I,2} + u[0]) \end{bmatrix}.$$

The condition $x[1] = 0$ implies $x_{I,1} = -u[0]$, $x_{I,2} = -u[0]$, hence all states that can be controlled to zero in one step are given by

$$x_I = \begin{bmatrix} -u[0] \\ -u[0] \\ x_{I,3} \end{bmatrix},$$

and this is a two-dimensional set. Note that this implies that the considered system has an eigenvalue at zero.

Question 2. *Consider the continuous-time system* $\dot{x} = Ax + Bu$*. Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(a) *Compute the reachability matrix R.*

(b) *Determine if the system is reachable.*

(c) *Compute the set of states that can be reached from the state,* $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution 2.

(a) The reachability matrix is

$$R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

(b) The rank$(R) = 1 < n = 2$, therefore the system is not reachable.

(c) Note that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{bmatrix} 0 \\ e^t \end{bmatrix} + \begin{bmatrix} \int_0^t e^{t-\tau}u(\tau)d\tau \\ 0 \end{bmatrix}.$$

Note that, by a proper selection of $u(\tau)$ in the interval $0 \leq \tau < t$ it is possible to *assign* $\int_0^t e^{t-\tau}u(\tau)d\tau$. Therefore, the states that can be reached at time t from x_0 are described by

$$x(t) = \begin{bmatrix} 0 \\ e^t \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

with $\lambda \in I\!R$.

Question 3. *Consider the discrete-time system* $x[k+1] = Ax[k] + Bu[k]$*. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(a) *Compute the reachability matrix R.*

(b) *Determine if the system is reachable.*

(c) *Compute the reachable subspaces in one step, two steps and three steps.*

Solution 3.

(a) The reachability matrix is

$$R = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

(b) The rank$(R) = 2 < n = 2$, therefore the system is not reachable.

(c) The set of reachable states in one step is

$$\mathcal{R}_1 = \text{span} B = \text{span} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The set of reachable states in two steps is

$$\mathcal{R}_2 = \text{span}[B, AB] = \text{span} \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The set of reachable states in three steps is

$$\mathcal{R}_3 = \text{span}[B, AB, A^2B] = \mathcal{R}_2.$$