Week 5 Tutorial

Question 1. *Consider the linear discrete-time system described by the equations*

$$x^+[k] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

For each of the following cases, determine the stability properties of the system and justify your conclusion:

(a) *When* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

(b) *When* $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

Solution 1.

(a) When

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues of A are obtained by solving $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

which gives:

$$(1-\lambda)^2 = 0$$

$$\lambda_{1,2} = 1$$

The two eigenvalues of A are located at $\lambda = 1$ *(The eigenvalues of a triangular matrix are the elements of the main diagonal of the triangular matrix).*

This implies that the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

The characteristic polynomial $\det(A - \lambda I) = 0$ is $(A - I)^2 = 0$, but the minimal polynomial is $(A - I) = 0$.

This means that the geometric multiplicity of $\lambda = 1$ in the minimal polynomial is equal to 1. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the unit circle must have a geometric multiplicity equal to 1.

Therefore, the equilibrium(s) of the system will be stable.

(b) When

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues are found by solving $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

Expanding the determinant:

$$(2-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

The eigenvalues are:

$$\lambda_{1,2} = 1$$

The two eigenvalues of A are the same part (a) and are located at $\lambda = 1$.

This again implies that the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

The characteristic polynomial $\det(A - \lambda I) = 0$ is still $(A - I)^2 = 0$, However, in this case, the characteristic polynomial is also the minimal polynomial of A.

This is because $(A - I) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$

This means that the geometric multiplicity of $\lambda = 1$ in the minimal polynomial is equal to 2. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the unit circle must have a geometric multiplicity equal to 1.

Therefore, the equilibrium(s) of the system will be unstable.

Question 2. *Consider the linear continuous-time system described by the equations*

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_1$$
$$\dot{x}_3 = -x_1$$

What can you conclude about the stability of the system?

Solution 2.

(a) This can be expressed in state-space form as

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The stability of the system is determined by the eigenvalues of A, which are found by solving

$$\det(A - \lambda I) = 0.$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ -1 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = (-1-\lambda)\lambda^2 = 0..$$

Solving for λ, we find the eigenvalues:

$$\lambda_1 = -1, \quad \lambda_{2,3} = 0.$$

- The presence of an eigenvalue at $\lambda_1 = -1$ suggests exponential decay for at least one mode.
- The two eigenvalues at $\lambda_{2,3} = 0$ indicate that certain states may not decay to zero, implying that the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

  The characteristic polynomial $\det(A - \lambda I) = 0$ is $(-I - A)A^2 = 0$, but the minimal polynomial is $(A - I)A = 0$.

  This is because $(-I - A)A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

  This means that the geometric multiplicity of $\lambda_{2,3} = 0$ in the minimal polynomial is equal to 1. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the imaginary axis must have a geometric multiplicity equal to 1.

  Therefore, the equilibrium(s) of the system will be stable.

Question 3. *Consider the linear continuous-time system described by the equations*

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

For each of the following cases, determine the stability properties of the system and justify your conclusion:

(a) When $A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

(b) When $A = \begin{bmatrix} -3 & 4 & -4 \\ 0 & 5 & -1 \\ 0 & 4 & -7 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

Solution 3.

(a) When

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

To determine the stability properties of the system, we need to analyse the eigenvalues of A which are obtained by solving $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 0 & 0 \\ -1 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = (-\lambda) \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0$$

Thus, the eigenvalues are:

$$\lambda_{1,2,3} = 0$$

Since all eigenvalues are exactly zero, the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

The characteristic polynomial $\det(A - \lambda I) = 0$ is still $A^3 = 0$; however, in this case, the minimal polynomial of A is A^2. This is because $A^2 = 0$

This means that the geometric multiplicity of $\lambda_{1,2,3} = 0$ in the minimal polynomial is equal to 2. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the imaginary axis must have a geometric multiplicity equal to 1.

Therefore, the equilibrium(s) of the system will be unstable.

(b) When

$$A = \begin{bmatrix} -3 & 4 & -4 \\ 0 & 5 & -1 \\ 0 & 4 & -7 \end{bmatrix}$$

The characteristic equation is given by $\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -3-\lambda & 4 & -4 \\ 0 & 5-\lambda & -1 \\ 0 & 4 & -7-\lambda \end{bmatrix}\right) = (-3-\lambda) \begin{vmatrix} 5-\lambda & -1 \\ 4 & -7-\lambda \end{vmatrix} = (-3-\lambda)(5-\lambda)(-7-\lambda)$$

$$(5-\lambda)(-7-\lambda) - (-1)(4) = \lambda^2 + 2\lambda - 31 = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = -3, \quad \lambda_1 = 5, \quad \lambda_1 = -7.$$

The eigenvalue $\lambda_1 = 5$ is positive, meaning the system contains an exponentially growing mode. Therefore, since at least one eigenvalue has a positive real part, the equilibrium(s) of the system will be unstable.