Week 4 Tutorial

Question 1. *Consider the linear continuous-time system from last week described by the equations*

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + \alpha x_2 + u(t)\\ \dot{x}_2(t) &= x_1(t) + x_2(t) - \alpha x_2(t)\\ y(t) &= x_1(t)\end{aligned}$$

with $\alpha \in \mathbb{R}$ and constant, $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$.

1. *Study the stability properties of the system when $\alpha > 2$ (Hint: the Routh array maybe helpful).*
2. *Let $u(t) = -ky(t)$, where k is a constant. Write the equations of the closed-loop system and determine conditions on α and k such that the closed-loop system is asymptotically stable.*

Solution 1.

1. The characteristic polynomial of the matrix A is

$$\begin{aligned}\det(I\lambda - A) &= \det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}1 & \alpha\\ 1 & 1-\alpha\end{bmatrix}\right) = \det\left(\begin{bmatrix}\lambda - 1 & -\alpha\\ -1 & \lambda + \alpha - 1\end{bmatrix}\right)\\ &= (\lambda - 1)(\lambda + \alpha - 1) - \alpha\\ &= \lambda^2 + \lambda\alpha - \lambda - \lambda - \alpha + 1 - \alpha\\ &= \lambda^2 + \lambda(\alpha - 2) + (1 - 2\alpha)\end{aligned}$$

The Routh array is:

λ^2	1	$(1-2\alpha)$
λ^1	$(\alpha-2)$	0
λ^0	$(1-2\alpha)$	0

For $\alpha > 2$ there will be a sign change in the first column of the Routh array.

Therefore, one of the roots has to have a positive real part so the system is unstable.

2. The equations for the closed-loop system are

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y = Cx(t) + Du(t),$$

where

$$D = \begin{bmatrix}0\\ 0\end{bmatrix}, \quad \text{and} \quad u(t) = -Ky(t) = -k(Cx(t) + Du(t)) = -kCx(t).$$

Therefore

$$\dot{x}(t) = (A - kBC)x(t), \quad y = Cx(t),$$

with

$$A - kBC = \begin{bmatrix} 1 & \alpha \\ 1 & 1-\alpha \end{bmatrix} - k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-k & \alpha \\ 1 & 1-\alpha \end{bmatrix}.$$

The characteristic polynomial of matrix $A - kBC$ is

$$\begin{aligned} \det\left(I\lambda - (A - kBC)\right) &= \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1-k & \alpha \\ 1 & 1-\alpha \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} \lambda - 1 + k & -\alpha \\ -1 & \lambda + \alpha - 1 \end{bmatrix} \right) \\ &= (\lambda - 1 + k)(\lambda + \alpha - 1) - \alpha \\ &= \lambda^2 + \lambda\alpha - \lambda - \lambda - \alpha + 1 + k\lambda + k\alpha - k - \alpha \\ &= \lambda^2 + \lambda(\alpha - 2 + k) + (1 + k\alpha - k - 2\alpha) \end{aligned}$$

The Routh array is:

λ^2	1	$(1 + k\alpha - k - 2\alpha)$
λ^1	$(\alpha - 2 + k)$	0
λ^0	$(1 + k\alpha - k - 2\alpha)$	0

Therefore, the closed-loop system is asymptotically stable for all k and α if

$$\alpha - 2 + k > 0, \quad 1 + k\alpha - k - 2\alpha > 0.$$

where there are no sign changes in the first column of the Routh array.

Question 2. *Consider the discrete-time system* $x_{k+1} = Ax_k$.

1. *Let*

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Consider the initial state

$$x[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and plot $x[k]$ *on the state space for* $k = 1, 2, 3, 4$. *Exploiting the obtained result discuss the stability of the equilibrium* $x_e = 0$.

2. *Let*

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Consider the initial state

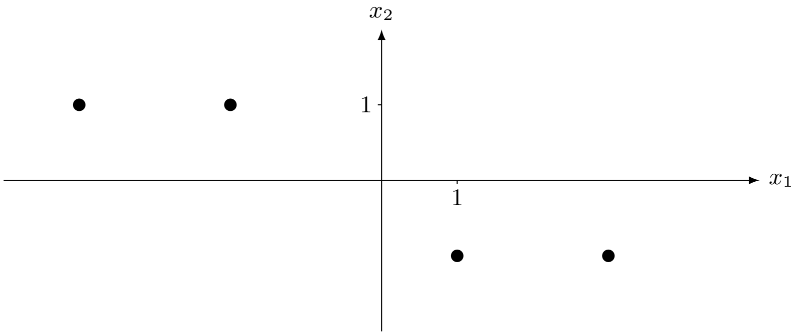
$$x[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and plot $x[k]$ *on the state space for* $k = 1, 2, 3, 4$. *Exploiting the obtained result discuss the stability of the equilibrium* $x_e = 0$.

Solution 2.

1. Note that

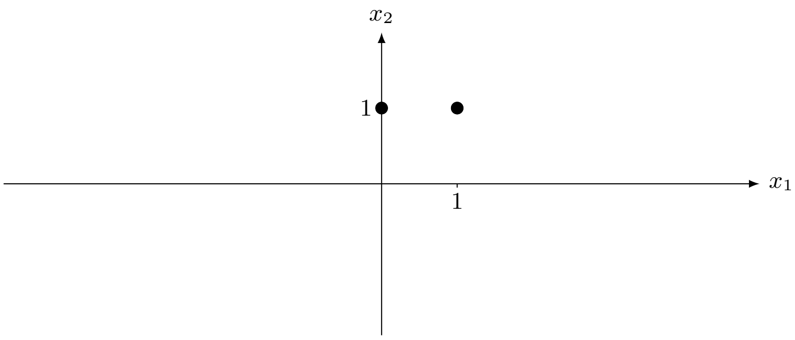
$$x[1] = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad x[2] = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad x[3] = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad x[4] = \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$



This implies that the equilibrium $x = 0$ is unstable. (Note that to decide the instability of an equilibrium, it is enough that one trajectory does not satisfy the "$\epsilon - \delta$" argument in the definition of stability.)

2. Note that

$$x[1] = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x[2] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad x[3] = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x[4] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



This trajectory is such that the "$\epsilon - \delta$" argument holds, however we cannot conclude stability of the equilibrium $x = 0$ only from properties of one trajectory.

Question 3. *Consider the discrete-time system*

$$x_{k+1} = Ax_k = \begin{bmatrix} 1 & 1 \\ a & -1 \end{bmatrix} x_k$$

with $a \in \mathbb{R}$.

Show that the system is asymptotically stable for $-2 < a < 0$, *and it is unstable for* $a < -2$ *and* $a > 0$.

Solution 3. The characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = (\lambda - 1)(\lambda + 1) - a = \lambda^2 - 1 - a.$$

Hence the eigenvalues of A are

$$\lambda_1 = +\sqrt{1+a} \qquad \lambda_2 = -\sqrt{1+a}.$$

The system is asymptotically stable if (and only if)

$$|\lambda_1| < 1 \qquad |\lambda_2| < 1.$$

Observe that λ_1 and λ_2 are real if $a \geq -1$ and are imaginary if $a < -1$. Moreover, $|\lambda_1| = |\lambda_2|$ and this is smaller than one if (and only if) $-2 < a < 0$.