Week 3 Tutorial

Question 1. *Consider a linear continuous-time system described by the equations*

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + \alpha x_2 + u(t)\\ \dot{x}_2(t) &= x_1(t) + x_2(t) - \alpha x_2(t)\\ y(t) &= x_1(t)\end{aligned}$$

with $\alpha \in \mathbb{R}$ *and constant,* $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$ *and* $u(t) \in \mathbb{R}$.

1. *Let* $u(t) = 0$*, for all* $t \geq 0$*. Compute the equilibrium points of the system as a function of* α.
2. *Assume now* $u(t) = u(0)$*, for all* $t \geq 0$*, where* $u(t) \neq 0$*. Compute the equilibrium points of the system as a function of* α.
3. *Discuss similarities and differences between the results in part (a) and part (b).*

Solution 1.

1. To begin with, note that

$$A = \begin{bmatrix} 1 & \alpha \\ 1 & 1-\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and that $\det(A) = 1 - 2\alpha$. If $1 - 2\alpha \neq 0$, the matrix A is invertible. Hence the only equilibrium for $u(t) = u(0) = 0$ is $x(0) = -A^{-1}Bu(0) = 0$ for all $t \geq 0$. If $\alpha = 1/2$ and $\det(A) = 0$, then to find the equilibrium points, we need to solve the equations $\dot{x}_1(t) = \dot{x}_2(t) = 0$ for $u(t) = 0$, that is

$$0 = x_1(t) + \frac{1}{2}x_2(t) \qquad 0 = x_1(t) + x_2(t) - \frac{1}{2}x_2(t) = x_1(t) + \frac{1}{2}x_2(t)$$

This means that all equilibrium points described by

$$x(t) = x(0) = \delta \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

where δ can be any number. Thus, when $\alpha = 1/2$ and $u(t) = u(0) = 0$, the system has infinitely many equilibrium points on a straight line.

2. As for part (a), if $1 - 2\alpha \neq 0$, the matrix A is invertible hence the only equilibrium for $u(t) = u(0) \neq 0$ is $x(0) = -A^{-1}Bu(0) = 0$ for all $t \geq 0$. That is

$$x(0) = \frac{u(0)}{1-2\alpha}\begin{bmatrix} 1-\alpha & -\alpha \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{u(0)}{1-2\alpha}\begin{bmatrix} 1-\alpha \\ -1 \end{bmatrix} \text{ or } \frac{u(0)}{2\alpha - 1}\begin{bmatrix} \alpha - 1 \\ 1 \end{bmatrix}$$

If $\alpha = 1/2$ and $\det(A) = 0$, then to find the equilibrium points, we need to solve the equations $\dot{x}_1(t) = \dot{x}_2(t) = 0$ for $u(t) = u(t) \neq 0$, that is

$$0 = x_1(t) + \frac{1}{2}x_2(t) + u(0) \qquad 0 = x_1(t) + \frac{1}{2}x_2(t)$$

These equations do not have any solution for $u(0) \neq 0$; that is, the system does not have any equilibrium points.

3. If the matrix A is invertible, regardless of the value of the input signal, the system has one equilibrium point. If A is not invertible, the existence of equilibrium points depends upon the value of the input signal. If $u(t) = u(0) = 0$, there are infinitely many equilibria, whereas if $u(t) = u(0) \neq 0$, there are no equilibria.

Question 2. *An ideal op-amp circuit is given in Figure 1*

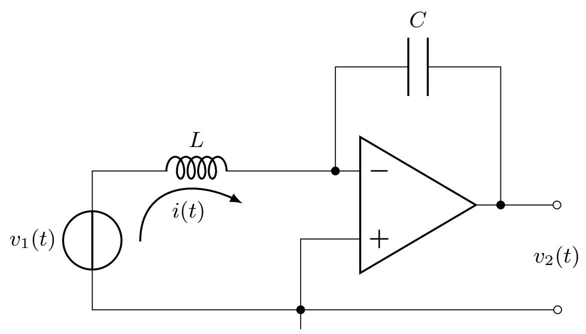

Figure 1

where $i(t)$ is the current, $v_1(t)$ is the input and $v_2(t)$ is the output.

1. *Derive the state space model for the circuit in Figure 1 using the state variables $x_1 = i(t)$ and $x_2 = v_2(t)$.*
2. *Using your answer from part 1, obtain the transfer function $G(s)$ of the circuit in Figure 1.*
3. *Find the state transition matrix e^{At} such that $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$.*
4. *Find the equilibrium point of the circuit in Figure 1.*

Solution 2.

1. First define v_1 and v_2

$$v_1 = v_L$$
$$v_1 = L\frac{di_L}{dt} = L\frac{di}{dt} \quad \therefore \quad \frac{di}{dt} = \frac{v_1}{L}$$
$$v_2 = v_C = \frac{1}{C}\int i_C dt \quad \therefore \quad \frac{dv_2}{dt} = \frac{i_C}{C} = -\frac{i}{C}$$

Define the derivatives of the state variable $\dot{x}_1$ and $\dot{x}_2$

$$x_1 = i \qquad\qquad x_2 = v_2$$

$$\dot{x}_1 = \frac{di}{dt} = \frac{v_1}{L} \qquad\qquad \dot{x}_2 = \frac{dv_2}{dt} = -\frac{i}{C} = -\frac{1}{C}x_1$$

Define the input u and output y

$$y = v_2 = x_2$$

$$u = v_1 \quad \therefore \quad \dot{x}_1 = \frac{v_1}{L} = \frac{1}{L}u$$

Therefore, the state space model is

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}\,.$$

2. The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}\,.$$

Therefore

$$\begin{aligned} G(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ \frac{1}{C} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} + \mathbf{0} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 \\ -\frac{1}{Cs^2} & \frac{1}{s} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{Ls} \\ -\frac{1}{LCs^2} \end{bmatrix} \\ &= -\frac{1}{LCs^2} \end{aligned}$$

Note that you get the same answer using the impedances of the components in the Laplace domain.

$$T(s) = \frac{V_2(s)}{V_1(s)} = -\frac{Z_2}{Z_1} = -\frac{1/Cs}{Ls} = -\frac{1}{LCs^2}$$

which you would have learnt in your previous module on classical control.

3. Method 1

Recall the definition of e^{At}.

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

First, calculate

$$A^2 = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore $A^3 = \mathbf{0}$, $A^4 = \mathbf{0}$, and so on.

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix} t = \begin{bmatrix} 1 & 0 \\ -\frac{1}{C} & 1 \end{bmatrix}$$

Method 2
We find e^{At} by integrating both $\dot{x}_1(t)$ and $\dot{x}_2(t)$

$$\dot{x}_1(t) = 0 \qquad \therefore \quad x_1(t) = x_1(0)$$

$$\dot{x}_2(t) = -\frac{1}{C}x_1(t) \qquad \therefore \quad x_2(t) = -\frac{t}{C}x_1(0) + x_2(0)$$

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) \quad \therefore \quad e^{At} = \begin{bmatrix} 1 & 0 \\ -\frac{t}{C} & 1 \end{bmatrix} \quad \boxed{\mathbf{1}}$$

Method 3
An alternative way to find e^{At}, which is not covered in this module, is to use the formula $e^{At} = \mathscr{L}^{-1}\{[sI - A]^{-1}\}$

$$e^{At} = \mathscr{L}^{-1}\left\{ \begin{bmatrix} s & 0 \\ \frac{1}{C} & s \end{bmatrix}^{-1} \right\} = \mathscr{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s} & 0 \\ -\frac{1}{Cs^2} & \frac{1}{s} \end{bmatrix} \right\} = \begin{bmatrix} 1 & 0 \\ -\frac{t}{C} & 1 \end{bmatrix}$$

4. To begin with, note that $\det(A) = 0$ meaning the matrix A is not invertible. Therefore, to find the equilibrium points, we need to solve the equations $\dot{x}_1(t) = \dot{x}_2(t) = 0$, that is

$$\dot{x}_1(t) = \frac{1}{L}u = 0 \therefore u = 0 \quad \text{and} \quad \dot{x}_2(t) = -\frac{1}{C}x_1(t) = 0 \therefore x_1(t) = 0$$

This means that all equilibrium points described by

$$x(t) = x(0) = \delta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \iff u(t) = 0$$

where δ can be any number. Thus, when $u(t) = u(0) = 0$, the system has infinitely many equilibrium points on a straight line, and when $u(t) = u(0) \neq 0$, the system has no equilibria.