Week 2 Tutorial

Question 1. *Consider the translational mechanical system shown in Figure 1, where $y_1(t)$ and $y_2(t)$ denote the displacements of the associated masses from their static equilibrium positions, and $f(t)$ represents a force applied to the first mass m_1.*

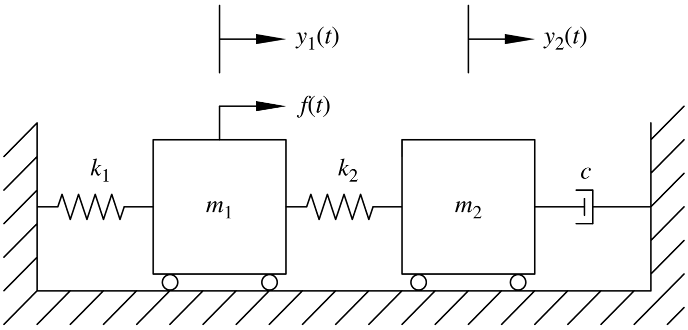

Figure 1: Translational mechanical system.

The system's parameters are the masses m_1 and m_2, viscous damping coefficient c, and spring stiffnesses k_1 and k_2. The input is the applied force $u(t) = f(t)$, and the outputs are taken as the mass displacements.

Derive a valid state-space realization for the mechanical system. That is, specify the state variables and derive the coefficient matrices A, B, C, and D.

Solution 1.

1. Newton's second law applied to each mass yields:

$$m_1\ddot{y}_1(t) + k_1 y_1(t) - k_2[y_2(t) - y_1(t)] = f(t),$$

$$m_2\ddot{y}_2(t) + c\dot{y}_2(t) + k_2[y_2(t) - y_1(t)] = 0.$$

   Define the state variables:

$$x_1(t) = y_1(t), \quad x_2(t) = y_2(t) - y_1(t), \quad x_3(t) = \dot{y}_1(t), \quad x_4(t) = \dot{y}_2(t).$$

   The state-space representation is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -\frac{k_1}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ 0 & -\frac{k_2}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t),$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t).$$

Question 2. *Consider the electrical network shown in Figure 2. The two inputs are the independent voltage and current sources $v_{in}(t)$ and $i_{in}(t)$, and the single output is the inductor voltage $v_L(t)$.*

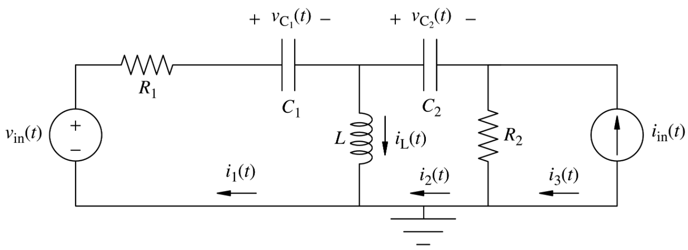

Figure 2: Electrical Circuit.

Using clockwise circulating mesh currents $i_1(t)$, $i_2(t)$, and $i_3(t)$, Kirchhoff's voltage and current laws yield:

$$R_1 i_1(t) + v_{C_1}(t) + L\frac{d}{dt}[i_1(t) - i_2(t)] = v_{in}(t),$$

$$L\frac{d}{dt}[i_2(t) - i_1(t)] + v_{C_2}(t) + R_2[i_2(t) - i_3(t)] = 0,$$

$$i_3(t) = -i_{in}(t),$$

$$i_L(t) = i_1(t) - i_2(t).$$

Derive a valid state-space realization for the electrical network. That is, specify the state variables and derive the coefficient matrices A, B, C, and D.

Solution 2.

1. Define the state variables:

$$x_1(t) = v_{C_1}(t), \quad x_2(t) = v_{C_2}(t), \quad x_3(t) = i_L(t),$$

and inputs:

$$u_1(t) = v_{\text{in}}(t), \quad u_2(t) = i_{\text{in}}(t).$$

2. State equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{(R_1+R_2)C_1} & -\frac{1}{(R_1+R_2)C_1} & \frac{R_2}{(R_1+R_2)C_1} \\ -\frac{1}{(R_1+R_2)C_2} & -\frac{1}{(R_1+R_2)C_2} & -\frac{R_1}{(R_1+R_2)C_2} \\ -\frac{R_2}{(R_1+R_2)L} & \frac{R_1}{(R_1+R_2)L} & -\frac{R_1R_2}{(R_1+R_2)L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{(R_1+R_2)C_1} & -\frac{R_2}{(R_1+R_2)C_1} \\ \frac{1}{(R_1+R_2)C_2} & -\frac{R_2}{(R_1+R_2)C_2} \\ \frac{R_2}{(R_1+R_2)L} & \frac{R_1R_2}{(R_1+R_2)L} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

3. Output equation:

$$y(t) = \begin{bmatrix} -\frac{R_2}{(R_1+R_2)} & \frac{R_1}{(R_1+R_2)} & -\frac{R_1R_2}{(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{R_2}{(R_1+R_2)} & \frac{R_1R_2}{(R_1+R_2)} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Question 3. *Consider the matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

1. *Compute the characteristic polynomial of A and find the eigenvalues of A.*
2. *Compute three linearly independent eigenvectors for A.*
3. *Find a similarity transformation L such that $\hat{A} = L^{-1}AL$ is a diagonal matrix.*
4. *Compute e^{At} as a function of t.*
5. *Compute $\sin(At)$ as a function of t.*

Solution 3.

1. Compute the characteristic polynomial:

$$\det(sI - A) = \det \begin{bmatrix} s-2 & 0 & 0 \\ -1 & s-1 & 0 \\ 0 & 0 & s-3 \end{bmatrix}.$$

Expanding along the third row:

$$\det(sI - A) = (s-3) \times \det \begin{bmatrix} s-2 & 0 \\ -1 & s-1 \end{bmatrix}.$$

Compute the determinant of the 2×2 matrix:

$$\det \begin{bmatrix} s-2 & 0 \\ -1 & s-1 \end{bmatrix} = (s-2)(s-1) - 0 = s^2 - 3s + 2$$

Thus:

$$\det(sI - A) = (s-3)(s^2 - 3s + 2).$$

Factorizing:

$$\det(sI - A) = (s-3)(s-2)(s-1).$$

Hence, the eigenvalues are $\lambda = 3, 2, 1$.

2. Solve $Av = \lambda v$ for each eigenvalue λ:
For $\lambda = 3$:

$$(A - 3I)v = 0 \implies \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0.$$

A solution is $v_1 = [0, 0, 1]^\top$.

For $\lambda = 2$:

$$(A - 2I)v = 0 \implies \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} v = 0.$$

A solution is $v_2 = [1, 1, 0]^\top$.
For $\lambda = 1$:

$$(A - I)v = 0 \implies \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} v = 0.$$

A solution is $v_3 = [0, 1, 0]^\top$.

3. Let $M = [v_1, v_2, v_3]$ be the matrix of eigenvectors:

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that

$$AM = M\Lambda, \implies A = M\Lambda M^{-1},$$

hence, $L = M$ and $\hat{A} = \Lambda$ where

$$\hat{A} = \Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Compute e^{At}:

$$e^{At} = Me^{\hat{A}t}M^{-1},$$

where

$$e^{\hat{A}t} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}.$$

To compute M^{-1}, first find the determinant of M:

$$\det(M) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

$$\det(M) = 0 - 1(-1) + 0 = 1.$$

Since $\det(M) = 1$, the cofactor matrix is:

$$\text{Cofactor matrix } = \text{adj}(M) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus the inverse matrix is:

$$M^{-1} = \operatorname{adj}(M) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Now compute $Me^{\hat{A}t}$:

$$Me^{\hat{A}t} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} = \begin{bmatrix} 0 & e^{2t} & 0 \\ 0 & e^{2t} & e^{t} \\ e^{3t} & 0 & 0 \end{bmatrix}.$$

Finally:

$$e^{At} = Me^{\hat{A}t}M^{-1} = \begin{bmatrix} 0 & e^{2t} & 0 \\ 0 & e^{2t} & e^{t} \\ e^{3t} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 & 0 \\ e^{2t} - e^{t} & e^{t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

5. Compute $\sin(At)$:

$$\sin(At) = M\sin(\hat{A}t)M^{-1},$$

where

$$\sin(\hat{A}t) = \begin{bmatrix} \sin(3t) & 0 & 0 \\ 0 & \sin(2t) & 0 \\ 0 & 0 & \sin(t) \end{bmatrix}.$$

Now compute $M\sin(\hat{A}t)$:

$$M\sin(\hat{A}t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sin(3t) & 0 & 0 \\ 0 & \sin(2t) & 0 \\ 0 & 0 & \sin(t) \end{bmatrix} = \begin{bmatrix} 0 & \sin(2t) & 0 \\ 0 & \sin(2t) & \sin(t) \\ \sin(3t) & 0 & 0 \end{bmatrix}$$

Finally:

$$e^{At} = M\sin(\hat{A}t)M^{-1} = \begin{bmatrix} 0 & \sin(2t) & 0 \\ 0 & \sin(2t) & \sin(t) \\ \sin(3t) & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \sin(2t) & 0 & 0 \\ \sin(2t) - \sin(t) & \sin(t) & 0 \\ 0 & 0 & \sin(3t) \end{bmatrix}.$$