Week 12 Tutorial - Example Exam Question

The final exam for 'Advanced Control Systems' consists of 4 questions, each worth 25 marks. Below is an example question of the type you can expect in the exam.

Question 1. **[25 marks]**

A student's knowledge is modelled using a discrete-time system where the student's knowledge level evolves over time based on their study effort and natural forgetting. The system is described by the following equations

$$M_1[k+1] = M_1[k] - \beta_1 M_1[k] + \gamma E[k] - \alpha M_1[k],$$

$$M_2[k+1] = M_2[k] - \beta_2 M_2[k] + \alpha M_1[k] + \delta M_2[k],$$

where $M_1[k]$ and $M_2[k]$ represent the student's short-term and long-term knowledge, respectively, with the signal $E[k]$ being the student's study effort.

The constants β_1 and β_2 are the forgetting rates for short-term and long-term knowledge, respectively. The constant α represents how effectively short-term knowledge transfers to long-term memory, γ measures how efficiently studying contributes to short-term knowledge, and δ accounts for the reinforcement of long-term memory through review.

The student's study effort is adjusted based on the gap between their perceived knowledge level and their target knowledge goal M_0, modelled as:

$$E[k+1] = \eta(M_0 - M_1[k] - M_2[k]),$$

where η is a parameter governing how strongly the student reacts to their knowledge gap.

(a) Derive the state space model for the student's knowledge using the state variables $x_1 = M_1$, $x_2 = M_2$ and $x_3 = E$, where the target knowledge goal is the input $u = M_0$, and the output is the sum of knowledge, $y = M_1 + M_2$. That is, derive the coefficient matrices A, B, C, and D.

[4 marks]

(b) Assume that the constants for the system are defined as follows: $\beta_1 = 1$, $\beta_2 = 1$, $\alpha = 1$, $\gamma = 0$, $\delta = 2$ and $\eta = 1$.

(i) Determine and justify whether the system is reachable, controllable, and observable.

[6 marks]

(ii) Assume that $u[k] = u[0]$, for all $k \geq 0$, where $u[0] \neq 0$. Compute the equilibrium points of the system.

[5 marks]

(iii) Using the PBH test, determine the unobservable modes of the system.

[5 marks]

(iv) Determine if the system is stable in terms of Lyapunov stability, and if the system is unstable, comment on whether the system is stabilizable using feedback control.

[5 marks]

Solution 1.

(a) We can define the state vector as:

$$\mathbf{x}[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix} = \begin{bmatrix} M_1[k] \\ M_2[k] \\ E[k] \end{bmatrix}$$

From the given system equations, we can express the system in state-space form. We start with the discrete-time system equations:

$$x_1[k+1] = (1 - \beta_1 + \alpha)x_1[k] + \gamma x_3[k]$$

$$x_2[k+1] = \alpha x_1[k] + (1 - \beta_2 + \delta)x_2[k]$$

$$x_3[k+1] = -\eta x_1[k] - \eta x_2[k] + \eta u$$

These equations can be rewritten in matrix form as:

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + Bu[k]$$

where the system matrices A, B, C, and D are derived as follows.

First, express the system in matrix form:

$$\mathbf{x}[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \end{bmatrix} = \begin{bmatrix} 1 - (\beta_1 + \alpha) & 0 & \gamma \\ \alpha & 1 - \beta_2 + \delta & 0 \\ -\eta & -\eta & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \eta \end{bmatrix} u[k]$$

Thus, the state-space matrices are:

$$A = \begin{bmatrix} 1 - (\beta_1 + \alpha) & 0 & \gamma \\ \alpha & 1 - \beta_2 + \delta & 0 \\ -\eta & -\eta & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \eta \end{bmatrix}$$

The output is given by:

$$y[k] = C\mathbf{x}[k] + Du[k]$$

where $y[k] = x_1[k] + x_2[k]$, so:

$$C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad D = 0$$

Thus, the state-space model is:

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + Bu[k]$$

$$y[k] = C\mathbf{x}[k]$$

(b) (i) To determine if the system is reachable, controllable, and observable, we need to analyze the rank of the reachability and observability matrices.

Assuming that the constants for the system are defined as follows: $\beta_1 = 1$, $\beta_2 = 1$, $\alpha = 1$, $\gamma = 0$, $\delta = 2$ and $\eta = 1$, the state-space matrices are:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The reachability matrix is given by:

$$\mathcal{R} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

In this case, the rank of $\mathcal{R}$ is equal to 1, so the system is not reachable.
In terms of controllability, since it is a discrete time system, we need to calculate A^3

$$A^2 = A \times A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

Therefore

$$A^3 = A^2 \times A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 8 & 0 \\ -2 & -4 & 0 \end{bmatrix}$$

Note that

$$\text{Im}A^3 \not\subset \text{Im}R$$

Alternative method for controllability

The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s+1 & 0 & 0 & 0 \\ -1 & s-2 & 0 & 0 \\ 1 & 1 & s & 1 \end{array}\right]$$

Note that A is a lower triangular matrix and, therefore, the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 0$.
The reachability pencil loses rank at $s = -1$ and $s = 2$, indicating the presence of two unreachable modes at $s = -1$ and $s = 2$. As a result, the system is **non-reachable** and also **not controllable**, as the unreachable modes are not at $s = 0$.

The observability matrix is given by:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

In this case, the rank of $\mathcal{O}$ is equal to 2, so the system is not observable.

(ii) If $\det(I - A) = \neq 0$, then the matrix $(A - I)$ is invertible and the only equilibrium for $u[k] = u[0] \neq 0$ would be $x[0] = (I - A)^{-1}Bu[0]$ for all $t \geq 0$ (discrete-time system).

In this case $\det(I - A) = -2$ and therefore

$$x[0] = -\begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[0] = \begin{bmatrix} 0.5 & 0 & 0 \\ -0.5 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[0] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[0]$$

Thus, the system has one equilibrium point.

Alternative method for controllability

To find the equilibrium points, we need to solve the equations $x_1[k+1] = x_1[k]$, $x_2[k+1] = x_2[k]$ and $x_3[k+1] = x_3[k]$ for $u[k] \neq 0$, that is

$$\begin{aligned} x_1[k] &= -x_1[k] \implies x_1[k] = 0 \\ x_2[k] &= x_1[k] + 2x_2[k] \implies x_2[k] = -x_1[k] \implies x_2[k] = 0 \\ x_3[k] &= -x_1[k] - x_2[k] + u[0] \implies x_3[k] = u[0] \end{aligned}$$

This means that all equilibrium points described by

$$x[k] = x[0] = \begin{bmatrix} 0 \\ 0 \\ u[0] \end{bmatrix},$$

Thus, the system has one equilibrium point.

(iii) To determine the unobservable modes using the PBH test, we need to compute the observability pencil

$$\begin{bmatrix} sI - A \\ \hline C \end{bmatrix} = \begin{bmatrix} s+1 & 0 & 0 \\ -1 & s-2 & 0 \\ 1 & 1 & s \\ \hline 1 & 1 & 0 \end{bmatrix}$$

and for each eigenvalue $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 0$ of the matrix A, we need to check the rank.

In this case, the observability pencil loses rank for $\lambda_3 = 0$, indicating the presence of one unobservable mode at $s = 0$. The rank of the observability matrix was 2, so we know there is only one unobservable mode.

(iv) To show that the system is unstable, we analyze the eigenvalues of the matrix A, which are $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 0$.

If any eigenvalue has a magnitude greater than 1, the system is unstable. Therefore, the eigenvalue $\lambda_2 = 2$ makes the system unstable. (Note that $\lambda_1 = -1$ and $\lambda_3 = 0$ are simple roots, so although these eigenvalue prevent asymptotic stability, they do not make the system unstable).

To determine whether the system can be stabilized using feedback control, we need to determine if $s = 2$ is reachable.

The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s+1 & 0 & 0 & 0 \\ -1 & s-2 & 0 & 0 \\ 1 & 1 & s & 1 \end{array}\right]$$

The reachability pencil loses rank at $s = -1$ and $s = 2$, indicating the presence of two unreachable modes at $s = -1$ and $s = 2$. Therefore, this system cannot be made stable using feedback.