Week 11 Tutorial

Question 1. *Consider the discrete-time system*

$$x_{k+1} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_k,$$
$$y_k = \begin{bmatrix} 3 & -1 \end{bmatrix} x_k.$$

(a) Show that the system is observable.

(b) Design an asymptotic observer where both eigenvalues of $A + LC$ are zero and with state $\hat{x}_k$, such that $e_k = x_k - \hat{x}_k = 0$ for all $k \geq N$. Determine the smallest value of N for which the above condition can be satisfied.

(c) Let $u_k = K\hat{x}_k + v_k$ with $K = [3/4, 3/4]$.
Write the equations of the closed-loop system, with state $[x_k, \hat{x}_k]$, input v_k and output y_k, and determine the eigenvalues of this system.

Solution 1.

(a) The observability matrix is

$$O = \begin{bmatrix} 3 & -1 \\ 0 & -5 \end{bmatrix},$$

which has rank equal to two. The system is, therefore, observable.

(b) An asymptotic observer is described by

$$\sigma\xi = F\xi + Ly + Hu$$
$$\xi_{k+1} = (A - LC)\xi_k + Ly_k + Bu_k$$
$$\xi_{k+1} = (A + LC)\xi_k - Ly_k + Bu_k$$

for some $L = [L_1 \ L_2]'$, where ξ_k is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. To obtain L should be such that both eigenvalues of $A + LC$ are zero. Note that

$$A + LC = \begin{bmatrix} 1 + 3L_1 & -2 - L_1 \\ 3 + 3L_2 & -1 - L_2 \end{bmatrix},$$

and

$$\det(sI - (A + LC)) = s^2 + s(L_2 - 3L_1) + 5(1 + L_2).$$

Hence,

$$L_2 - 3L_1 = 0\,, \qquad 1 + L_2 = 0$$

Therefore,

$$L_1 = -1/3\,, \qquad L_2 = -1.$$

Recall that

$$\begin{aligned}
\sigma e &= (A + LC)e \\
e_k &= (A + LC)e_{k-1} \\
e[1] &= (A + LC)e[0] \\
e[2] &= (A + LC)e[1] = (A + LC)^2 e[0] \\
e[3] &= (A + LC)e[2] = (A + LC)^3 e[0] \\
&\vdots
\end{aligned}$$

It is is easy to see that $e_k = (A + LC)^k e_0$ therefore to find the smallest value of N such that $e_k = 0$ for all $k \geq N$, we need to find when the smallest value of k such that $(A + LC)^k = 0$ as we know that $e[0] \neq 0$.

With this selection of L we have $(A + LC)^2 = 0$, hence $N = 2$.

(c) By the separation principle the eigenvalues of the closed-loop system are the eigenvalues of the observer and of the matrix

$$A + BK = \begin{bmatrix} 7/4 & -5/4 \\ 9/4 & -7/4 \end{bmatrix}.$$

Hence, the eigenvalues of the closed-loop system are $\{1/2, -1/2, 0, 0\}$.

Question 2. *Consider the continuous-time system*

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 3 & -1 \end{bmatrix} x
\end{aligned}$$

(a) Design an asymptotic observer with a double pole at -6.

(b) Suppose x_0 *is the observer state evaluated in part 1. Let* $u = Kx_0 + v$ *with* $K = [-4, 2]$. *Compute the eigenvalues of the closed-loop system.*

Solution 2.

(a) An asymptotic observer is described by

$$\dot{\xi} = (A + LC)\xi - Ly + Bu$$

for some $L = [L_1 \; L_2]'$, where ξ is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. Note that

$$A + LC = \begin{bmatrix} 1 + 3L_1 & -2 - L_1 \\ 3 + 3L_2 & -1 - L_2 \end{bmatrix},$$

and its characteristic polynomial is

$$s^2 + s(L_2 - 3L_1) + 5(1 + L_2).$$

This should be equal to

$$(s+6)^2 = s^2 + 12s + 36.$$

As a result,

$$L_1 = -29/15 \qquad L_2 = 31/5.$$

(b) By the separation principle the eigenvalues of the closed-loop system are the eigenvalues of the observer and of the matrix

$$A + BK = \begin{bmatrix} -3 & 0 \\ 7 & -3 \end{bmatrix}.$$

Hence, the eigenvalues of the closed-loop system are $\{-3, -3, -6, -6\}$.

Question 3. *Consider the continuous-time system $\dot{x} = Ax$, $y = Cx$. Let*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(a) Show, using PBH test, that the system is observable.

(b) Design an asymptotic observer for the system. Select the output injection gain L such that the matrix $A + LC$ has two eigenvalues equal to -3.

(c) Suppose that one can measure $y(t)$ and a delayed copy of $y(t)$ given by $y(t-\tau)$, with $\tau > 0$.

For $t \geq \tau$, express the vector $Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix}$ from $x(0)$.

Show that the relation determined above can be used, for any $\tau > 0$, to compute $x(0)$ as a function of $Y(t)$, where $t \geq \tau$. Argue that the above result can be used to determine $x(t)$ from $Y(t)$, for $t \geq \tau$, exactly.

Solution 3.

(a) The observability pencil is

$$\left[\begin{array}{cc} s & -1 \\ 0 & s+2 \\ \hline 1 & 0 \end{array}\right],$$

which has rank two for any s. Hence, the system is observable.

(b) An asymptotic observer is described by

$$\dot{\xi} = (A + LC)\xi - Ly$$

for some $L = [L_1 \; L_2]'$, where ξ is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. Note that

$$A + LC = \begin{bmatrix} L_1 & 1 \\ L_2 & -2 \end{bmatrix}$$

and its characteristic polynomial is

$$s^2 + s(2 - L_1) - 2L_1 - L_2.$$

This should be equal to $(s+3)^2 = s^2 + 6s + 9$, yielding

$$L_1 = -4 \qquad L_2 = -1.$$

(c) Note that

$$y(t) = Ce^{At}x(0)$$

and replacing t with $t - \tau$ one has

$$y(t-\tau) = Ce^{A(t-\tau)}x(0).$$

Then, for $t \geq \tau$,

$$Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix} = \begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} e^{At}x(0).$$

For the given A and C let's calculate $e^{-A\tau}$ explicitly

$$e^{-A\tau} = \begin{bmatrix} 1 & \tau - \frac{2\tau^2}{2} + \frac{4\tau^2}{3} - \frac{4\tau^2}{8} \cdots \\ 0 & 1 - 2\tau + \frac{4\tau^2}{2} - \frac{8\tau^3}{3} + \frac{16\tau^4}{4} \cdots \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} - \frac{1}{2}(1 - 2\tau + \frac{4\tau^2}{2} - \frac{8\tau^2}{3} + \frac{16\tau^2}{8} \cdots) \\ 0 & 1 - 2\tau + \frac{4\tau^2}{2} - \frac{8\tau^3}{3} + \frac{16\tau^4}{4} \cdots \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{1}{2} - \frac{e^{2\tau}}{2} \\ 0 & e^{2\tau} \end{bmatrix}$$

where

$$Ce^{-A\tau} = \begin{bmatrix} 1 & \frac{1}{2} - \frac{e^{2\tau}}{2} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{2\tau}-1}{2} \end{bmatrix}$$

which is invertible for all $\tau > 0$. Hence

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{2\tau}-1}{2} \end{bmatrix}^{-1} Y(t).$$

The above relation implies that, for all $t \geq \tau$, it is possible to obtain exactly $x(t)$.