Week 10 Tutorial

Question 1. *Consider the continuous-time system $\dot{x} = Ax + Bu$. Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find a matrix K such that $P_\lambda(A + BK) = \{-1, -2\}$. Solve the problem in two ways:

(a) using Ackermann's formula (explained the 'module notes'), which states

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} R^{-1} p(A),$$

such that the characteristic polynomial of $A + BK$ is equal to $p(s)$.

(b) using a direct computation seen in the lecture, i.e. without computing the reachability matrix of the system.

Solution 1.

(a) The general theory states that the state feedback is given by

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} R^{-1} p(A),$$

where R is the reachability matrix and $p(s)$ is the desired closed-loop characteristic polynomial, in this case $p(s) = (s+1)(s+2) = s^2 + 3s + 2$. As a result,

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} \left(\frac{1}{4} \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} 12 & 16 \\ 24 & 36 \end{bmatrix} = -\begin{bmatrix} 3 & 5 \end{bmatrix},$$

yielding

$$A + BK = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix},$$

which has eigenvalues equal to -1 and -2, as requested.

(b) Let

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

and note that

$$A + BK = \begin{bmatrix} 1 + K_1 & 2 + K_2 \\ 3 + K_1 & 4 + K_2 \end{bmatrix}.$$

The characteristic polynomial of $A + BK$ is

$$\det(sI - (A + BK)) = s^2 + s(-5 - K_1 - K_2) + (-2K_2 + 2K_1 - 2),$$

and this should be equal to $p(s) = s^2 + 3s + 2$. As a result, K_1 and K_2 should be such that

$$-5 - K_1 - K_2 = 3 \qquad -2K_2 + 2K_1 - 2 = 2,$$

which yields $K_1 = -3$ and $K_2 = -5$. Note that, because the system has only one input and it is reachable, the state feedback assigning the eigenvalues is unique.

Question 2. *Consider the continuous-time system*

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & -1 \end{bmatrix} x.\end{aligned}$$

(a) Show that the system is controllable and observable.

(b) Design a state feedback control law

$$u = Kx + v$$

such that the closed-loop system has two eigenvalues at -3.

Solution 2.

(a) The reachability matrix is

$$R = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix},$$

which is full rank, hence the system is reachable and controllable.
The observability matrix is

$$O = \begin{bmatrix} 3 & -1 \\ 0 & -5 \end{bmatrix},$$

which is full rank, hence the system is observable.

(b) Let

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

and note that

$$A + BK = \begin{bmatrix} 1 + K_1 & -2 + K_2 \\ 3 + K_1 & -1 - K_2 \end{bmatrix}.$$

The characteristic polynomial of $A + BK$ is

$$\det(sI - (A + BK)) = s^2 + s(-K_1 + K_2) + (-3K_1 + 5 - 4K_2),$$

and this should be equal to $p(s) = (s+3)^2 = s^2 + 6s + 9$. As a result, K_1 and K_2 should be such that

$$-K_1 + K_2 = 6 \qquad - 3K_1 + 5 - 4K_2 = 9,$$

which yields $K_1 = -4$ and $K_2 = 2$.

Question 3. *Consider the continuous-time system*

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 3 & -1 + \epsilon \\ 1 & 2 - \epsilon \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 1 \end{bmatrix} x.\end{aligned}$$

(a) Show that the system is controllable for any $\epsilon \neq 1$. Study the stabilisability of the system for $\epsilon = 1$.

(b) Show that the system is observable for any $\epsilon \neq 1/2$. Study the detectability of the system for $\epsilon = 1/2$.

(c) Assume $\epsilon = 0$. Design a state feedback control law

$$u = Kx + v$$

such that the closed-loop system has two eigenvalues equal to -2.

(d) Show that the state feedback control law designed above stabilizes the system for any $\epsilon \in (-4, 1/7)$.

Solution 3.

(a) The reachability matrix is

$$R = \begin{bmatrix} 0 & \epsilon - 1 \\ 1 & 2 - \epsilon \end{bmatrix},$$

and $\det(R) = 1 - \epsilon$. As a result the system is reachable and controllable for $\epsilon \neq 1$. Let $\epsilon = 1$ and consider the reachability pencil

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{cc|c} s-3 & 0 & 0 \\ -1 & s-1 & 1 \end{array}\right],$$

which has rank equal to one for $s = 3$. The system is therefore not stabilizable. Note that it is possible to obtain this conclusion without computing the reachability pencil. In fact, for $\epsilon = 1$ the eigenvalues of A are $\{3, 1\}$, hence if there is an unreachable mode this is associated to a value of s with positive real part.

(b) The observability matrix is

$$O = \begin{bmatrix} -1 & 1 \\ -2 & -2\epsilon + 3 \end{bmatrix},$$

and $\det(O) = 2\epsilon - 1$. As a result the system is observable for $\epsilon \neq 1/2$. Let $\epsilon = 1/2$ and consider the observability pencil

$$\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \left[\begin{array}{cc} s-3 & 1/2 \\ -1 & s-3/2 \\ \hline -1 & 1 \end{array}\right].$$

Because the system is not observable, this matrix has to have rank equal to one for some s. To find such an s consider the submatrix

$$\left[\begin{array}{cc} -1 & s-3/2 \\ \hline -1 & 1 \end{array}\right].$$

Its determinant is $s - 5/2$, hence the unobservable mode is $s = 5/2$ and the system is not detectable. Note that it is possible to obtain this conclusion without computing the observability pencil. In fact, for $\epsilon = 1/2$ the eigenvalues of A are $\{5/2, 2\}$, hence if there is an unobservable mode this is associated to a value of s with positive real part.

(c) If $\epsilon = 0$ we have

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}.$$

Setting $K = [K_1,\ K_2]$ yields

$$\det(sI - (A + BK)) = s^2 + s(-5 - K_2) + (K_1 + 3K_2 + 7),$$

which should be equal to $(s+2)^2$. This is achieved setting

$$K_1 = 24 \qquad\qquad K_2 = -9.$$

(d) Consider now the matrix

$$A + BK = \begin{bmatrix} 3 & \epsilon - 1 \\ 25 & -7 - \epsilon \end{bmatrix}.$$

Its characteristic polynomial is

$$\det(sI - (A + BK)) = s^2 + s(4 + \epsilon) + (4 - 28\epsilon),$$

which has both roots with negative real part (by Routh test) if and only if $\epsilon \in (-4, 1/7)$.