Tutorial Problem Sheet 6

Positioning Relative to the Mid-Semester Quiz

This problem sheet is intended to develop the core structural ideas that underpin the in-person Mid-Semester Quiz. In particular, it builds fluency with:

- State-space modelling and interpretation of (A, B),
- Equilibria and stability classification (continuous and discrete time),
- Reachability and controllability analysis,
- Structural reasoning and justification of results.

The Mid-Semester Quiz will require clear mathematical reasoning, concise justification, and the ability to combine ideas across topics when analysing unseen systems. The questions here are designed to strengthen conceptual understanding and prepare you for the Mid-Semester Quiz.

Question 1.

Consider the translational mechanical system described by the following differential equations,

$$m_1\ddot{y}_1(t) + c\dot{y}_1(t) - k[y_2(t) - y_1(t)] = 0 \ ,$$

$$m_2\ddot{y}_2(t) + k[y_2(t) - y_1(t)] = u(t) \ .$$

Given that the state variables are defined as

$$x_1(t) = y_1(t), \quad x_2(t) = y_2(t) - y_1(t), \quad x_3(t) = \dot{y}_1(t), \quad x_4(t) = \dot{y}_2(t).$$

Derive the state-space realisation for the mechanical system, giving the A and B matrices explicitly.

Show your workings.

Solution 1.

The state-space equations are:

$$\dot{x}_1 = \dot{y}_1 = x_3,$$

$$\dot{x}_2 = \frac{d}{dt}(y_2 - y_1) = \dot{y}_2 - \dot{y}_1 = x_4 - x_3.$$

Since

$$\ddot{y}_1 = \frac{k}{m_1}(y_2 - y_1) - \frac{c}{m_1}\dot{y}_1,$$

Therefore

$$\dot{x}_3 = \frac{k}{m_1}x_2 - \frac{c}{m_1}x_3$$

Since

$$\ddot{y}_2 = -\frac{k}{m_2}(y_2 - y_1) + \frac{1}{m_2}u,$$

Therefore

$$\dot{x}_4 = -\frac{k}{m_2}x_2 + \frac{1}{m_2}u$$

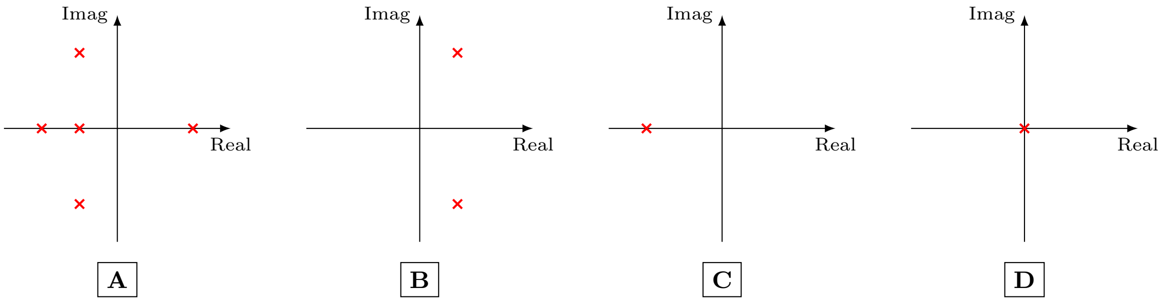
Therefore, the state-space representation is:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & \frac{k}{m_1} & -\frac{c}{m_1} & 0 \\ 0 & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t).$$

Question 2.

Consider the following Argand diagrams for four linear continuous-time systems are given, where the red crosses denote the eigenvalue locations of each system.



Can you match each of the linear continuous-time systems given to their correct stability characteristics?

(a) Unstable

(b) Stable

(c) Asymptotically Stable

(d) Insufficient Information

Justify your answers.

Solution 2.

The matching is:

- System A: (a) Unstable. There is an eigenvalue in the right-half plane ($\Re\{\lambda\} > 0$), hence an exponentially growing mode.
- System B: (a) Unstable. The complex conjugate pair lies in the right-half plane ($\Re\{\lambda\} > 0$), giving exponentially growing oscillations.
- System C: (c) Asymptotically stable. The eigenvalue shown lies strictly in the left-half plane ($\Re\{\lambda\} < 0$), so all modes decay.
- System D: (d) Insufficient Information: Stable or Unstable. The plot shows an eigenvalue at the origin, so the system is *not* asymptotically stable. Whether it is (Lyapunov) stable or unstable depends on the *minimal polynomial multiplicity* of the eigenvalue $\lambda = 0$:
  - if $\lambda = 0$ has minimal polynomial multiplicity 1 (i.e. it is semisimple), the corresponding mode is constant and the system can be stable (marginally stable);
  - if $\lambda = 0$ has minimal polynomial multiplicity > 1, solutions contain polynomial terms (e.g. $t, t^2, \dots$) and the system is unstable.

  Thus, with only the eigenvalue location shown, we can conclude "stable or unstable".

Question 3.

Consider the following linear continuous-time systems $\dot{x}(t) = Ax(t)+Bu(t)$ where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

Which statement best describes the internal properties of the system in terms of Lyapunov stability?

(a) Unstable

(b) Stable

(c) Asymptotically Stable

Justify your answer.

Solution 3.

To determine Lyapunov stability we examine the eigenvalues of A (internal stability of $\dot{x} = Ax$). Compute the characteristic polynomial:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda+3 & 0 \\ -2 & 0 & \lambda+1 \end{bmatrix}.$$

Expanding along the second row (only the middle entry is nonzero),

$$\det(\lambda I - A) = (\lambda+3) \det \begin{bmatrix} \lambda & 0 \\ -2 & \lambda+1 \end{bmatrix} = (\lambda+3)\big(\lambda(\lambda+1) - 0\big) = (\lambda+3)\lambda(\lambda+1).$$

Hence the eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = -1, \qquad \lambda_3 = -3.$$

There are no eigenvalues with positive real part, and the only eigenvalue on the imaginary axis is $\lambda = 0$. Since $\lambda = 0$ is a simple root of the characteristic polynomial (algebraic multiplicity 1), it is semisimple, so the origin is *Lyapunov stable* but *not* asymptotically stable.

Therefore the correct statement is: (b) Stable.

Question 4.

Consider the linear discrete-time system described by the following equations,

$$x_1^+ = 3x_1 + x_2 + u \ ,$$
$$x_2^+ = 0 \ .$$

Which statement best describes the input-to-state properties of the system?

(a) Reachable and controllable

(b) Reachable but not controllable

(c) Controllable but not reachable

(d) Not reachable and not controllable

Justify your answer.

Solution 4.

Write the system as

$$x^+ = Ax + Bu, \qquad A = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Reachability. From the second equation,

$$x_2^+ = 0 \ \Rightarrow \ x_2(k) = 0 \text{ for all } k \geq 1,$$

in particular starting from $x(0) = 0$ we have $x_2(k) = 0$ for all k. Hence states with $x_2 \neq 0$ can never be reached from the origin, so the system is *not reachable.*

Equivalently, the reachability matrix is

$$\mathcal{R} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \qquad \text{rank}(\mathcal{R}) = 1 < 2,$$

so the system is not reachable.

Controllability Given any initial state $x(0) = (x_1(0), x_2(0))$, choose at $k = 0$:

$$u(0) = -3x_1(0) - x_2(0).$$

Then

$$x_1(1) = 3x_1(0) + x_2(0) + u(0) = 0, \qquad x_2(1) = 0,$$

so $x(1) = 0$. Therefore the system is (c) Controllable but not reachable (in one step).

Alternatively (method 2), To answer whether the system is controllable, we could compute A^2 and its image.

$$A^2 = \begin{bmatrix} 9 & 3 \\ 0 & 0 \end{bmatrix}$$

The image of A^2 consists of the span of its columns:

$$\text{span}\left(\begin{bmatrix} 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right)$$

Again, these columns are multiples of each other, meaning the span is:

$$\text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

which is the same subspace as the image of R.

Since the column space of A^2 is contained within the column space of R, we can conclude:

$$\text{Im}(A^2) \subseteq \text{Im}(R)$$

This means the system is controllable because the image of A^2 spans the same dimensional space as $\mathcal{R}$

Alternatively (method 3), Check if the unreachable modes at zero. Another way to see why controllability-to-zero still holds is that the only unreachable state component is x_2, and its associated mode is at the origin:

$$x_2^+ = 0,$$

so the unreachable mode is already 'deadbeat' (its eigenvalue is 0) and is driven to zero in one step regardless of the input. You could also use the PBH test to check this.

Thus the system is (c) Controllable but not reachable.

Question 5.

What are the unique properties of linear discrete-time systems that give rise to the fact that controllability $\not\Longrightarrow$ reachability for linear discrete-time systems? Please select all the correct answers.

(a) Linear discrete-time systems are not always reservable in time

(b) Linear discrete-time systems are able to reach the origin in finite time

(c) A needs to be of full rank for controllability $\Longrightarrow$ reachability

(d) A must have no eigenvalues at $\lambda = 1$ for controllability $\Longrightarrow$ reachability

Justify your answer for each statement.

Solution 5.

For linear discrete-time systems

$$x_{k+1} = Ax_k + Bu_k,$$

controllability (to the origin) and reachability (from the origin) are not equivalent in general. The key issue is that A may be singular.

(a) **TRUE.** Linear discrete-time systems are not always reversible in time. If A is singular, the map $x_{k+1} = Ax_k$ is not invertible, so past states cannot in general be uniquely reconstructed. This lack of time-reversibility is what allows a system to be controllable to the origin without being reachable from the origin.

(b) **TRUE.** Linear discrete-time systems are able to reach the origin in finite time. Unlike continuous-time systems (which require exponential decay), discrete-time systems can be driven exactly to the origin in a finite number of steps. In particular, unreachable modes associated with eigenvalues at $\lambda = 0$ go to zero in finite time, allowing controllability-to-zero even when reachability fails.

(c) **TRUE.** A needs to be of full rank for controllability $\Longrightarrow$ reachability. If A is nonsingular, the system is time-reversible and controllability to the origin implies reachability from the origin. When A is not full rank (i.e. singular), the implication can fail.

(d) **FALSE.** A must have no eigenvalues at $\lambda = 1$ for controllability $\Longrightarrow$ reachability. The issue is not eigenvalues at $\lambda = 1$, but eigenvalues at $\lambda = 0$ (which make A singular). Eigenvalues at $\lambda = 1$ do not affect the equivalence between controllability and reachability in this way.

Question 6.

Consider the linear discrete-time system $x[k+1] = Ax[k] + Bu[k]$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Which statement best describes the internal properties of the system in terms of Lyapunov stability?

(a) Unstable

(b) Stable

(c) Asymptotically Stable

Justify your answer.

Solution 6.

Consider the autonomous system $x[k+1] = Ax[k]$ with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since A is diagonal, its eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = 0.$$

For discrete-time *Lyapunov stability* (stability, not asymptotic stability):

- all eigenvalues must satisfy $|\lambda_i| \leq 1$; and
- any eigenvalue with $|\lambda_i| = 1$ must have *minimal polynomial multiplicity* equal to 1 (i.e. it must be semisimple; no repeated factor for that eigenvalue in the minimal polynomial).

Here $|\lambda_i| \leq 1$ for all i. Moreover, because A is diagonal, the minimal polynomial is

$$m_A(\lambda) = \lambda(\lambda - 1),$$

so the factor $(\lambda - 1)$ appears with multiplicity 1 (even though $\lambda = 1$ has algebraic multiplicity 2). Hence the unit-circle eigenvalues do *not* generate polynomial growth, and the origin is Lyapunov stable.

However, asymptotic stability would require $|\lambda_i| < 1$ for all i, which is violated by $\lambda = 1$. Indeed,

$$x_1[k] = x_1[0], \qquad x_2[k] = x_2[0], \qquad x_3[k] = 0 \ (k \geq 1),$$

so trajectories remain bounded but do not converge to the origin in general.

Therefore, the system is (b) Stable but not asymptotically stable.

Question 7.

Which of the following statements are true? Please select all the true statements.

(a) If a system is reachable, then all of its equilibria must be stable

(b) Reachability $\implies$ controllability for both discrete and continuous time systems

(c) The state-transition matrix, $e^A t$ is always invertible

(d) For linear discrete-time systems, the states you can reach in n steps may be different from the states you can reach in $n+1$ steps *(where n is the number of state variables of the system)*

Justify your answer for each statement.

Solution 7.

(a) **FALSE.** Reachability is an input–state property (whether inputs can steer the state), whereas stability is an internal property of the autonomous dynamics (eigenvalues of A). A system can be reachable and still have an unstable equilibrium (e.g. choose any unstable A and a B such that (A, B) is reachable). Thus reachability does not force equilibria to be stable.

(b) **TRUE.** For both continuous- and discrete-time LTI systems, reachability from the origin implies controllability to the origin when controllability is defined in the standard sense of being able to steer any initial state to any final state in finite time (in particular, to the origin). Intuitively, if every state can be reached from 0, then by concatenating trajectories (or using time-shifted inputs) one can also steer between arbitrary states, hence to 0.

(c) **TRUE.** The state-transition matrix for continuous time is e^{At}. It is always invertible, with inverse e^{-At}, since $e^{At}e^{-At} = e^{A(t-t)} = e^0 = I$. Hence e^{At} is nonsingular for all t.

(d) **FALSE.** The set of states reachable from the origin in exactly k steps is

$$\mathcal{R}_k = \operatorname{im}\begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix}.$$

By the Cayley–Hamilton theorem, A satisfies its characteristic equation

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0,$$

hence

$$A^n B + a_{n-1}A^{n-1}B + \cdots + a_1 AB + a_0 B = 0.$$

Therefore $A^n B$ is a linear combination of $B, AB, \ldots, A^{n-1}B$, so adding the $(n+1)$-step column block $A^n B$ cannot increase the reachable subspace:

$$\operatorname{im}\begin{bmatrix} B & AB & \cdots & A^n B \end{bmatrix} = \operatorname{im}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Thus, the set of states reachable in n and in $n+1$ steps is the same, so the statement is false.

Question 8.

Which of the following is the state transition matrix, e^{At}, when $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$?

Show your workings.

Solution 8.

The state transition matrix is

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Compute powers of A:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A^3 = 0.$$

Hence the series terminates:

$$e^{At} = I + At + \frac{A^2t^2}{2!}.$$

Now substitute:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad At = \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{A^2t^2}{2} = \begin{bmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Question 9.

Consider a linear continuous-time system described by the equations

$$\dot{x}_1(t) = x_1(t) + x_2(t) + u(t)$$
$$\dot{x}_2(t) = x_1(t) + x_2(t)$$
$$y(t) = 2x_1(t)$$

Which of the following statements is true about the system when $u(t) = 0$:

(a) The system has no equilibrium points

(b) The system has one unique equilibrium point

(c) The system has infinitely many equilibrium points

Justify your answer.

Solution 9.

Set $u(t) = 0$. An equilibrium point $x_e = \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}$ satisfies $\dot{x} = 0$, i.e.

$$0 = x_{1e} + x_{2e}, \qquad 0 = x_{1e} + x_{2e}.$$

These two equations are identical, so there is only one independent constraint:

$$x_{2e} = -x_{1e}.$$

Hence, the set of equilibria is the line

$$\left\{ x_e \in \mathbb{R}^2 : \ x_e = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}, \ \alpha \in \mathbb{R} \right\},$$

which contains infinitely many equilibrium points.

Therefore, the correct statement is: (c) The system has infinitely many equilibrium points.

Question 10.

Consider the linear discrete-time system described by the following equations,

$$x_1^+ = x_1 + u \ ,$$
$$x_2^+ = x_2 \ .$$

What state is reached from the initial state of $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in three steps when $u[k] = \alpha$ for all k.

Show your workings.

Solution 10.

Given

$$x_1[k+1] = x_1[k] + u[k], \qquad x_2[k+1] = x_2[k],$$

with $x[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $u[k] = \alpha$ for all k.

Compute step-by-step:

$$x_1[1] = x_1[0] + \alpha = 0 + \alpha = \alpha, \qquad x_2[1] = x_2[0] = 1,$$

$$x_1[2] = x_1[1] + \alpha = \alpha + \alpha = 2\alpha, \qquad x_2[2] = x_2[1] = 1,$$
$$x_1[3] = x_1[2] + \alpha = 2\alpha + \alpha = 3\alpha, \qquad x_2[3] = x_2[2] = 1.$$

Therefore,

$$x[3] = \begin{bmatrix} 3\alpha \\ 1 \end{bmatrix}.$$