

Tutorial Problem Sheet 6

Positioning Relative to the Mid-Semester Quiz

This problem sheet is intended to develop the core structural ideas that underpin the in-person Mid-Semester Quiz. In particular, it builds fluency with:

- State-space modelling and interpretation of (A, B) ,
- Equilibria and stability classification (continuous and discrete time),
- Reachability and controllability analysis,
- Structural reasoning and justification of results.

The Mid-Semester Quiz will require clear mathematical reasoning, concise justification, and the ability to combine ideas across topics when analysing unseen systems. The questions here are designed to strengthen conceptual understanding and prepare you for the Mid-Semester Quiz.

Question 1.

Consider the translational mechanical system described by the following differential equations,

$$m_1\ddot{y}_1(t) + c\dot{y}_1(t) - k[y_2(t) - y_1(t)] = 0 ,$$

$$m_2\ddot{y}_2(t) + k[y_2(t) - y_1(t)] = u(t) .$$

Given that the state variables are defined as

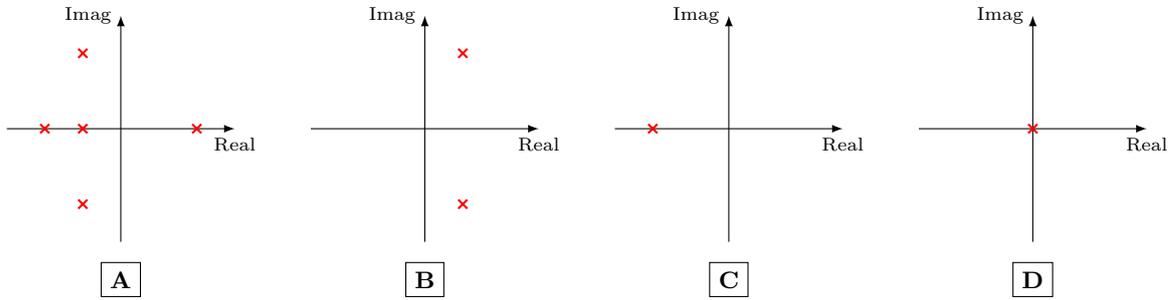
$$x_1(t) = y_1(t), \quad x_2(t) = y_2(t) - y_1(t), \quad x_3(t) = \dot{y}_1(t), \quad x_4(t) = \dot{y}_2(t).$$

Derive the state-space realisation for the mechanical system, giving the A and B matrices explicitly.

Show your workings.

Question 2.

Consider the following Argand diagrams for four linear continuous-time systems are given, where the red crosses denote the eigenvalue locations of each system.



Can you match each of the linear continuous-time systems given to their correct stability characteristics?

- (a) Unstable
- (b) Stable
- (c) Asymptotically Stable
- (d) Insufficient Information

Justify your answers.

Question 3.

Consider the following linear continuous-time systems $\dot{x}(t) = Ax(t) + Bu(t)$ where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

Which statement best describes the internal properties of the system in terms of Lyapunov stability?

- (a) Unstable
- (b) Stable
- (c) Asymptotically Stable

Justify your answer.

Question 4.

Consider the linear discrete-time system described by the following equations,

$$\begin{aligned}x_1^+ &= 3x_1 + x_2 + u, \\x_2^+ &= 0.\end{aligned}$$

Which statement best describes the input-to-state properties of the system?

- (a) Reachable and controllable
- (b) Reachable but not controllable
- (c) Controllable but not reachable
- (d) Not reachable and not controllable

Justify your answer.

Question 5.

What are the unique properties of linear discrete-time systems that give rise to the fact that controllability $\not\Rightarrow$ reachability for linear discrete-time systems? Please select all the correct answers.

- (a) Linear discrete-time systems are not always reservable in time
- (b) Linear discrete-time systems are able to reach the origin in finite time
- (c) A needs to be of full rank for controllability \Rightarrow reachability
- (d) A must have no eigenvalues at $\lambda = 1$ for controllability \Rightarrow reachability

Justify your answer for each statement.

Question 6.

Consider the linear discrete-time system $x[k + 1] = Ax[k] + Bu[k]$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Which statement best describes the internal properties of the system in terms of Lyapunov stability?

- (a) Unstable
- (b) Stable
- (c) Asymptotically Stable

Justify your answer.

Question 7.

Which of the following statements are true? Please select all the true statements.

- (a) If a system is reachable, then all of its equilibria must be stable
- (b) Reachability \implies controllability for both discrete and continuous time systems
- (c) The state-transition matrix, e^{At} is always invertible
- (d) For linear discrete-time systems, the states you can reach in n steps may be different from the states you can reach in $n + 1$ steps (*where n is the number of state variables of the system*)

Justify your answer for each statement.

Question 8.

Which of the following is the state transition matrix, e^{At} , when $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$?

Show your workings.

Question 9.

Consider a linear continuous-time system described by the equations

$$\dot{x}_1(t) = x_1(t) + x_2(t) + u(t)$$

$$\dot{x}_2(t) = x_1(t) + x_2(t)$$

$$y(t) = 2x_1(t)$$

Which of the following statements is true about the system when $u(t) = 0$:

- (a) The system has no equilibrium points
- (b) The system has one unique equilibrium point
- (c) The system has infinitely many equilibrium points

Justify your answer.

Question 10.

Consider the linear discrete-time system described by the following equations,

$$x_1^+ = x_1 + u ,$$

$$x_2^+ = x_2 .$$

What state is reached from the initial state of $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in three steps when $u[k] = \alpha$ for all k .

Show your workings.