Tutorial Problem Sheet 5

Question 1. *Consider the linear continuous-time system described by the equations*

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

For each of the following cases, determine the stability properties of the system and justify your conclusion:

(a) *When* $A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

(b) *When* $A = \begin{bmatrix} -3 & 4 & -4 \\ 0 & 5 & -1 \\ 0 & 4 & -7 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

Solution 1.

(a) When

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

To determine the stability properties of the system, we need to analyse the eigenvalues of A which are obtained by solving $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 0 & 0 \\ -1 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0$$

Thus, the eigenvalues are:

$$\lambda_{1,2,3} = 0$$

Since all eigenvalues are exactly zero, the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

The characteristic polynomial $\det(A - \lambda I) = 0$ is still $A^3 = 0$; however, in this case, the minimal polynomial of A is A^2. This is because $A^2 = 0$

This means that the geometric multiplicity of $\lambda_{1,2,3} = 0$ in the minimal polynomial is equal to 2. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the imaginary axis must have a geometric multiplicity equal to 1.

Therefore, the equilibrium(s) of the system will be <u>unstable</u>.

(b) When

$$A = \begin{bmatrix} -3 & 4 & -4 \\ 0 & 5 & -1 \\ 0 & 4 & -7 \end{bmatrix}$$

The characteristic equation is given by $\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -3-\lambda & 4 & -4 \\ 0 & 5-\lambda & -1 \\ 0 & 4 & -7-\lambda \end{bmatrix}\right) = (-3-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ 4 & -7-\lambda \end{vmatrix} = (-3-\lambda)\left[(5-\lambda)(-7-\lambda)+4\right]$$

$$= (-3-\lambda)(\lambda^2 + 2\lambda - 31) = 0.$$

Thus, the eigenvalues are:

$$\lambda_1 = -3, \quad \lambda_{2,3} = -1 \pm 4\sqrt{2}.$$

Since one eigenvalue satisfies $-1 + 4\sqrt{2} > 0$, the system contains an exponentially growing mode. Therefore, since at least one eigenvalue has a positive real part, the equilibrium(s) of the system will be unstable.

Question 2. *Consider the discrete-time system $x[k+1] = Ax[k] + Bu[k]$ from last week. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Using PHB test determine the unreachable modes.

(b) Show that the system is controllable.

Solution 2.

(a) The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s & -1 & 0 & 1 \\ 1 & s & 0 & -1 \\ 0 & 0 & s & 0 \end{array}\right].$$

This matrix has rank three for all $s \neq 0$, hence, the unreachable mode is $s = 0$.

(b) The system is controllable since the unreachable modes are at $s = 0$. Note that

$$A^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$\mathrm{Im} A^3 \subset \mathrm{Im} R,$$

which also proves the system is controllable.

Question 3. *Consider the continuous-time system $x[k+1] = Ax[k] + Bu[k]$ from last week. Let*

$$A = \begin{bmatrix} 0 & \gamma \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(a) Show using PHB test that the system is reachable.

(b) Express the state-space representation of the continuous-time system in controllable canonical form.

Solution 3.

(a) The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{cc|c} s & -\gamma & 1 \\ 1 & s & 0 \end{array}\right].$$

which has rank two for any s and any γ.

Hence, the system is always **reachable**.

(b) The controllable canonical form of a 2×2 matrix is given by

$$A_c = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

where α_0 and α_1 are the coefficients of the characteristic polynomial,

$$\lambda^2 + \alpha_1\lambda + \alpha_0 = \det(I\lambda - A) = \det \begin{bmatrix} \lambda & -\gamma \\ 1 & \lambda \end{bmatrix} = \lambda^2 + \gamma = 0$$

In this case $\alpha_1 = 0$ and $\alpha_0 = \gamma$. Therefore,

$$A_c = \begin{bmatrix} 0 & 1 \\ -\gamma & 0 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Question 4. *Consider the discrete-time system $x[k+1] = Ax[k] + Bu[k]$. Let*

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Using PHB test determine the unreachable modes.

(b) Comment on whether the system is controllable.

Solution 4.

(a) The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s+3 & -1 & 0 & 1 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s+2 & 0 \end{array}\right].$$

This matrix loses rank for $s = 1$ and $s = -2$, hence, these are unreachable modes of the system

(b) The system is not controllable since the unreachable modes are not at $s = 0$. Also note that

$$A^3 = \begin{bmatrix} -27 & 7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix},$$

hence

$$\mathrm{Im}A^3 \not\subset \mathrm{Im}R,$$

which proves the system is not controllable.