Tutorial Problem Sheet 4

Question 1. *Consider the linear discrete-time system described by the equations*

$$x^+[k] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

For each of the following cases, determine the stability properties of the system and justify your conclusion:

(a) *When* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

(b) *When* $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, *what can you conclude about the stability properties of the system?*

Solution 1.

(a) When

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues of A are obtained by solving $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

which gives:

$$(1-\lambda)^2 = 0$$

$$\lambda_{1,2} = 1$$

The two eigenvalues of A are located at $\lambda = 1$ *(The eigenvalues of a triangular matrix are the elements of the main diagonal of the triangular matrix).*

This implies that the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

The characteristic polynomial $\det(A - \lambda I) = 0$ is $(A - I)^2 = 0$, but the minimal polynomial is $(A - I) = 0$.

This means that the geometric multiplicity of $\lambda = 1$ in the minimal polynomial is equal to 1. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the unit circle must have a geometric multiplicity equal to 1.

Therefore, the equilibrium(s) of the system will be stable.

(b) When

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues are found by solving $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

Expanding the determinant:

$$(2-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

The eigenvalues are:

$$\lambda_{1,2} = 1$$

The two eigenvalues of A are the same part (a) and are located at $\lambda = 1$.

This again implies that the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

The characteristic polynomial $\det(A - \lambda I) = 0$ is still $(A - I)^2 = 0$, However, in this case, the characteristic polynomial is also the minimal polynomial of A.

This is because $(A - I) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$

This means that the geometric multiplicity of $\lambda = 1$ in the minimal polynomial is equal to 2. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the unit circle must have a geometric multiplicity equal to 1.

Therefore, the equilibrium(s) of the system will be unstable.

Question 2. *Consider the linear continuous-time system described by the equations*

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_1$$
$$\dot{x}_3 = -x_1$$

What can you conclude about the stability of the system?

Solution 2.

(a) This can be expressed in state-space form as

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The stability of the system is determined by the eigenvalues of A, which are found by solving

$$\det(A - \lambda I) = 0.$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ -1 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = (-1-\lambda)\lambda^2 = 0..$$

Solving for λ, we find the eigenvalues:

$$\lambda_1 = -1, \quad \lambda_{2,3} = 0.$$

- The presence of an eigenvalue at $\lambda_1 = -1$ suggests exponential decay for at least one mode.
- The two eigenvalues at $\lambda_{2,3} = 0$ indicate that certain states may not decay to zero, implying that the equilibrium(s) of the system are either stable (but not asymptotically stable) or unstable.

  The characteristic polynomial $\det(A - \lambda I) = 0$ is $(-I - A)A^2 = 0$, but the minimal polynomial is $(A - I)A = 0$.

  This is because $(-I - A)A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

  This means that the geometric multiplicity of $\lambda_{2,3} = 0$ in the minimal polynomial is equal to 1. For an equilibrium to be stable (but not asymptotically stable), all eigenvalues on the imaginary axis must have a geometric multiplicity equal to 1.

  Therefore, the equilibrium(s) of the system will be stable.

Question 3. *Consider the discrete-time system $x[k+1] = Ax[k] + Bu[k]$. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(a) *Compute the reachability matrix R.*

(b) *Determine if the system is reachable and compute the set of reachable states.*

(c) *Determine all states x_I such that $x[0] = x_I$ and $x[1] = 0$.*

Solution 3.

(a) The reachability matrix is

$$R = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 2 & -2 & -2 \end{bmatrix}.$$

(b) The first two columns of the reachability matrix are linearly independent and $\det(R) = 0$, hence, the system is not reachable. The set of reachable states is two-dimensional, and it is described by the linear combination of the first two columns of the reachable matrix.

(c) We have to determine all states which are controllable in one step. Instead of using the definition of controllable states in one step, let's perform a direct calculation. Let

$$x[0] = x_I = \begin{bmatrix} x_{I,1} \\ x_{I,2} \\ x_{I,3} \end{bmatrix}$$

and note that

$$x[1] = Ax[0] + Bu[0] = \begin{bmatrix} x_{I,2} + u[0] \\ -x_{I,1} - u[0] \\ 2(x_{I,2} + u[0]) \end{bmatrix}.$$

The condition $x[1] = 0$ implies $x_{I,1} = -u[0]$, $x_{I,2} = -u[0]$, hence all states that can be controlled to zero in one step are given by

$$x_I = \begin{bmatrix} -u[0] \\ -u[0] \\ x_{I,3} \end{bmatrix},$$

and this is a two-dimensional set. Note that this implies that the considered system has an eigenvalue at zero.

Question 4. *Consider the continuous-time system $\dot{x} = Ax + Bu$. Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(a) Compute the reachability matrix R.

(b) Determine if the system is reachable.

(c) Compute the set of states that can be reached from the state, $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution 4.

(a) The reachability matrix is

$$R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

(b) The $\text{rank}(R) = 1 < n = 2$, therefore the system is not reachable.

(c) Note that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{bmatrix} 0 \\ e^t \end{bmatrix} + \begin{bmatrix} \int_0^t e^{t-\tau}u(\tau)d\tau \\ 0 \end{bmatrix}.$$

Note that, by a proper selection of $u(\tau)$ in the interval $0 \leq \tau < t$ it is possible to *assign* $\int_0^t e^{t-\tau}u(\tau)d\tau$. Therefore, the states that can be reached at time t from x_0 are described by

$$x(t) = \begin{bmatrix} 0 \\ e^t \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

with $\lambda \in \mathbb{R}$.

Question 5. *Consider the discrete-time system $x[k+1] = Ax[k] + Bu[k]$. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Compute the reachability matrix R.

(b) Determine if the system is reachable.

(c) Compute the reachable subspaces in one step, two steps and three steps.

Solution 5.

(a) The reachability matrix is

$$R = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

(b) The $\text{rank}(R) = 2 < n = 2$, therefore the system is not reachable.

(c) The set of reachable states in one step is

$$\mathcal{R}_1 = \text{span}B = \text{span}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The set of reachable states in two steps is

$$\mathcal{R}_2 = \text{span}[B, AB] = \text{span}\begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} = \text{span}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The set of reachable states in three steps is

$$\mathcal{R}_3 = \text{span}[B, AB, A^2B] = \mathcal{R}_2.$$