Tutorial Problem Sheet 2

Question 1. *An ideal op-amp circuit is given in Figure 1*

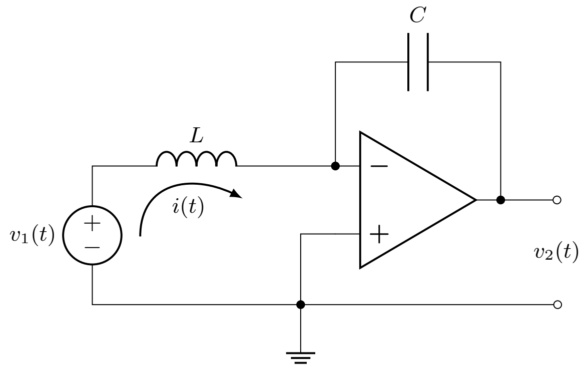

Figure 1

where $i(t)$ is the current, $v_1(t)$ is the input and $v_2(t)$ is the output.

(a) *Derive the state space model for the circuit in Figure 1 using the state variables $x_1 = i(t)$ and $x_2 = v_2(t)$.*

(b) *Using your answer from part (a), obtain the transfer function $G(s)$ of the circuit in Figure 1.*

(c) *Find the state transition matrix e^{At} such that $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$.*

(d) *Find the equilibrium point of the circuit in Figure 1.*

Solution 1.

(a) First define v_1 and v_2

$$v_1 = v_L$$
$$v_1 = L\frac{di_L}{dt} = L\frac{di}{dt} \quad \therefore \quad \frac{di}{dt} = \frac{v_1}{L}$$
$$v_2 = v_C = \frac{1}{C}\int i_C dt \quad \therefore \quad \frac{dv_2}{dt} = \frac{i_C}{C} = -\frac{i}{C}$$

Define the derivatives of the state variable $\dot{x}_1$ and $\dot{x}_2$

$$x_1 = i \qquad\qquad x_2 = v_2$$
$$\dot{x}_1 = \frac{di}{dt} = \frac{v_1}{L} \qquad\qquad \dot{x}_2 = \frac{dv_2}{dt} = -\frac{i}{C} = -\frac{1}{C}x_1$$

Define the input u and output y

$$y = v_2 = x_2$$
$$u = v_1 \quad \therefore \quad \dot{x}_1 = \frac{v_1}{L} = \frac{1}{L}u$$

Therefore, the state space model is

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}\,.$$

(b) The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}\,.$$

Therefore

$$\begin{aligned} G(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ \frac{1}{C} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} + \mathbf{0} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 \\ -\frac{1}{Cs^2} & \frac{1}{s} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{Ls} \\ -\frac{1}{LCs^2} \end{bmatrix} \\ &= -\frac{1}{LCs^2} \end{aligned}$$

Note that you get the same answer using the impedances of the components in the Laplace domain.

$$T(s) = \frac{V_2(s)}{V_1(s)} = -\frac{Z_2}{Z_1} = -\frac{1/Cs}{Ls} = -\frac{1}{LCs^2}$$

which you would have learnt in your previous module on classical control.

(c) Method 1

Recall the definition of e^{At}.

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

First, calculate

$$A^2 = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore $A^3 = \mathbf{0}$, $A^4 = \mathbf{0}$, and so on.

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix} t = \begin{bmatrix} 1 & 0 \\ -\frac{1}{C} & 1 \end{bmatrix}$$

Method 2

We find e^{At} by integrating both $\dot{x}_1(t)$ and $\dot{x}_2(t)$

$$\dot{x}_1(t) = 0 \qquad\qquad \dot{x}_2(t) = -\frac{1}{C}x_1(t)$$

$$\therefore \quad x_1(t) = x_1(0) \qquad\qquad \therefore \quad x_2(t) = -\frac{t}{C}x_1(0) + x_2(0)$$

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) \quad \therefore \quad e^{At} = \begin{bmatrix} 1 & 0 \\ -\frac{t}{C} & 1 \end{bmatrix} \quad \boxed{\mathbf{1}}$$

Method 3

An alternative way to find e^{At}, which is not covered in this module, is to use the formula $e^{At} = \mathscr{L}^{-1}\{[sI - A]^{-1}\}$

$$e^{At} = \mathscr{L}^{-1}\left\{ \begin{bmatrix} s & 0 \\ \frac{1}{C} & s \end{bmatrix}^{-1} \right\} = \mathscr{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s} & 0 \\ -\frac{1}{Cs^2} & \frac{1}{s} \end{bmatrix} \right\} = \begin{bmatrix} 1 & 0 \\ -\frac{t}{C} & 1 \end{bmatrix}$$

(d) To begin with, note that $\det(A) = 0$ meaning the matrix A is not invertible. Therefore, to find the equilibrium points, we need to solve the equations $\dot{x}_1(t) = \dot{x}_2(t) = 0$, that is

$$\dot{x}_1(t) = \frac{1}{L}u = 0 \therefore u = 0 \quad \text{and} \quad \dot{x}_2(t) = -\frac{1}{C}x_1(t) = 0 \therefore x_1(t) = 0$$

This means that all equilibrium points described by

$$x(t) = x(0) = \delta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \iff u(t) = 0$$

where δ can be any number. Thus, when $u(t) = u(0) = 0$, the system has infinitely many equilibrium points on a straight line, and when $u(t) = u(0) \neq 0$, the system has no equilibria.

Question 2. *Consider the discrete-time linear system*

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}u[k],$$
$$y[k] = \mathbf{C}\mathbf{x}[k],$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\frac{T}{C} & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{T}{L} \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

and $T > 0$ is the sampling period.

(a) *Show that the discrete-time state matrix $\mathbf{A}$ corresponds to the continuous-time system studied in Question 1.*

(b) *Compute $\mathbf{A}^k$ for all $k \geq 0$.*

(c) *Hence, derive an explicit expression for the state trajectory $\mathbf{x}[k]$ in terms of the initial condition $\mathbf{x}[0]$ and the input sequence $\{u[0], u[1], \ldots, u[k-1]\}$.*

(d) *Assume $u[k] = 0$ for all k. Describe the free response of the system.*

(e) *Determine all equilibrium points of the discrete-time system and compare them with the equilibrium points of the continuous-time model.*

Solution 2.

(a) For a continuous-time linear system

$$\dot{x}(t) = Ax(t),$$

the state trajectory is given by

$$x(t) = e^{At}x(0),$$

where the matrix exponential e^{At} is defined by the power series

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots .$$

In a discrete-time setting, the state is observed at sampling instants $t = kT$. The state transition over one sampling interval of length T is therefore

$$x((k+1)T) = e^{AT}x(kT).$$

This shows that the discrete-time state transition matrix is given by

$$A_d = e^{AT}.$$

For the system considered here,

$$A = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix}.$$

A direct computation gives

$$A^2 = \begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Since all higher powers of A are also zero, the series defining the matrix exponential terminates after the linear term:

$$e^{AT} = I + AT.$$

Hence,

$$e^{AT} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + T\begin{bmatrix} 0 & 0 \\ -\frac{1}{C} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{T}{C} & 1 \end{bmatrix}.$$

This explains why the discrete-time state matrix $\mathbf{A}$ appearing in this problem has the form $I + AT$. Such a structure is directly analogous to the continuous-time system studied previously, where $\dot{x}_2$ depends linearly on x_1 while $\dot{x}_1$ does not depend on the state.

(b) Method 1

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\frac{T}{C} & 1 \end{bmatrix}.$$

Compute successive powers:

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\frac{T}{C} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{T}{C} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2T}{C} & 1 \end{bmatrix}.$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\frac{2T}{C} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{T}{C} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{3T}{C} & 1 \end{bmatrix}.$$

$$\mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\frac{3T}{C} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{T}{C} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{4T}{C} & 1 \end{bmatrix}.$$

We observe the pattern: each multiplication by $\mathbf{A}$ leaves the diagonal entries equal to 1, keeps the upper-right entry equal to 0, and subtracts an additional $-\frac{T}{C}$ in the lower-left entry. Therefore, after k multiplications,

$$\mathbf{A}^k = \begin{bmatrix} 1 & 0 \\ -\frac{kT}{C} & 1 \end{bmatrix}, \qquad k \geq 0.$$

Method 2

Since

$$N^2 = \begin{bmatrix} 0 & 0 \\ -\frac{T}{C} & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where the matrix N is nilpotent. Therefore, using the binomial expansion,

$$\mathbf{A}^k = (I + N)^k = I + kN.$$

Hence

$$\mathbf{A}^k = \begin{bmatrix} 1 & 0 \\ -\frac{kT}{C} & 1 \end{bmatrix}, \qquad k \geq 0.$$

(c) From the general discrete-time solution formula,

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i}\mathbf{B}u[i].$$

Using the result from part (b),

$$\mathbf{A}^{k-1-i} = \begin{bmatrix} 1 & 0 \\ -\frac{(k-1-i)T}{C} & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{T}{L} \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{A}^{k-1-i}\mathbf{B} = \begin{bmatrix} \frac{T}{L} \\ -\frac{(k-1-i)T^2}{LC} \end{bmatrix}.$$

Therefore,

$$\mathbf{x}[k] = \begin{bmatrix} 1 & 0 \\ -\frac{kT}{C} & 1 \end{bmatrix} \mathbf{x}[0] + \sum_{i=0}^{k-1} \begin{bmatrix} \frac{T}{L} \\ -\frac{(k-1-i)T^2}{LC} \end{bmatrix} u[i].$$

(d) For $u[k] = 0$ for all k,

$$\mathbf{x}[k] = \mathbf{A}^k \mathbf{x}[0].$$

Hence,

$$x_1[k] = x_1[0], \qquad x_2[k] = x_2[0] - \frac{kT}{C} x_1[0].$$

Unless $x_1[0] = 0$, the second state grows linearly with k.

(e) An equilibrium $(\mathbf{x}_e, u_e)$ satisfies

$$\mathbf{x}_e = \mathbf{A}\mathbf{x}_e + \mathbf{B}u_e.$$

This gives

$$(I - \mathbf{A})\mathbf{x}_e = \mathbf{B}u_e.$$

Then

$$(I - \mathbf{A})\mathbf{x}_e = \begin{bmatrix} 0 & 0 \\ \frac{T}{C} & 0 \end{bmatrix} \begin{bmatrix} x_{1,e} \\ x_{2,e} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{T}{C} x_{1,e} \end{bmatrix}.$$

Similarly,

$$\mathbf{B}u_e = \begin{bmatrix} \frac{T}{L} u_e \\ 0 \end{bmatrix}.$$

Equating the two sides gives the system of equations

$$0 = \frac{T}{L} u_e, \qquad \frac{T}{C} x_{1,e} = 0.$$

Since $T > 0$, $L > 0$, and $C > 0$, these reduce to

$$u_e = 0, \qquad x_{1,e} = 0.$$

Equating both sides yields

$$u_e = 0, \qquad x_{1,e} = 0,$$

with no restriction on $x_{2,e}$. Hence, the equilibrium set is

$$u_e = 0, \qquad \mathbf{x}_e = \delta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \delta \in \mathbb{R}.$$

These equilibria are directly analogous to those of the continuous-time system: When the input is zero, there are infinitely many equilibrium points lying on a line in the state space.