Tutorial Problem Sheet 1

Question 1. *Consider the translational mechanical system shown in Figure 1, where $y_1(t)$ and $y_2(t)$ denote the displacements of the associated masses from their static equilibrium positions, and $f(t)$ represents a force applied to the first mass m_1.*

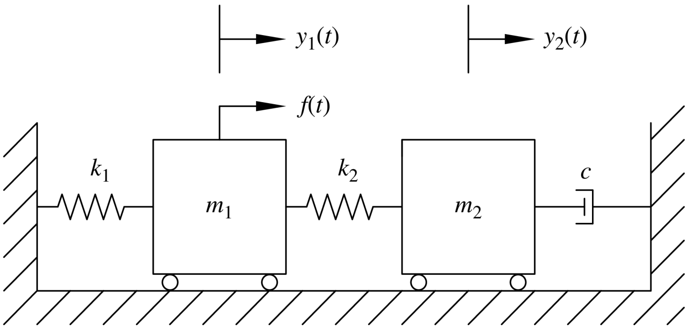

Figure 1: Translational mechanical system.

The system's parameters are the masses m_1 and m_2, the viscous damping coefficient c, and the spring stiffnesses k_1 and k_2. The input is the applied force $u(t) = f(t)$, and the outputs are taken as the mass displacements.

Derive a valid state-space realisation for the mechanical system. That is, specify the state variables and derive the coefficient matrices A, B, C, and D.

Solution 1.

(a) Newton's second law applied to each mass yields:

$$m_1\ddot{y}_1(t) + k_1 y_1(t) - k_2[y_2(t) - y_1(t)] = f(t),$$

$$m_2\ddot{y}_2(t) + c\dot{y}_2(t) + k_2[y_2(t) - y_1(t)] = 0.$$

Define the state variables:

$$x_1(t) = y_1(t), \quad x_2(t) = y_2(t) - y_1(t), \quad x_3(t) = \dot{y}_1(t), \quad x_4(t) = \dot{y}_2(t).$$

The state-space representation is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -\frac{k_1}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ 0 & -\frac{k_2}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t),$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t).$$

Question 2. *Consider the electrical network shown in Figure 2. The two inputs are the independent voltage and current sources $v_{in}(t)$ and $i_{in}(t)$, and the single output is the inductor voltage $v_L(t)$.*

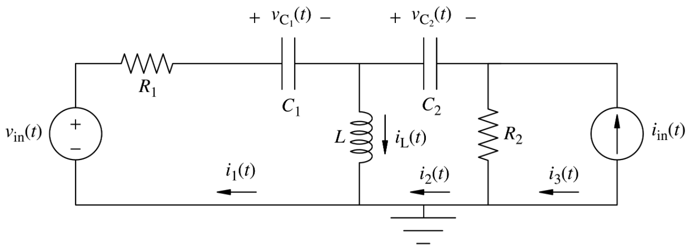

Figure 2: Electrical Circuit.

Using clockwise circulating mesh currents $i_1(t)$, $i_2(t)$, and $i_3(t)$, Kirchhoff's voltage and current laws yield:

$$R_1 i_1(t) + v_{C_1}(t) + L\frac{d}{dt}[i_1(t) - i_2(t)] = v_{in}(t),$$

$$L\frac{d}{dt}[i_2(t) - i_1(t)] + v_{C_2}(t) + R_2[i_2(t) - i_3(t)] = 0,$$

$$i_3(t) = -i_{in}(t), \qquad i_L(t) = i_1(t) - i_2(t).$$

Derive a valid state-space realisation for the electrical network. That is, specify the state variables and derive the coefficient matrices A, B, C, and D.

Solution 2.

(a) Define the state variables:

$$x_1(t) = v_{C_1}(t), \quad x_2(t) = v_{C_2}(t), \quad x_3(t) = i_L(t).$$

The inputs:

$$u_1(t) = v_{\text{in}}(t), \quad u_2(t) = i_{\text{in}}(t),$$

and the outputs:

$$y(t) = v_L(t) = L\dot{x}_3(t).$$

Using the relationship

$$R_1 i_1(t) + v_{C_1}(t) + L\frac{d}{dt}[i_1(t) - i_2(t)] = v_{\text{in}}(t),$$

can be written as

$$R_1 C_1 \dot{x}_1(t) + L\dot{x}_3(t) = -x_1(t) + u_1(t)\,. \tag{2.1}$$

Then using

$$L\frac{d}{dt}[i_2(t) - i_1(t)] + v_{C_2}(t) + R_2[i_2(t) - i_3(t)] = 0,$$

can be written as

$$R_2C_2\dot{x}_2(t) - L\dot{x}_3(t) = -x_2(t) - R_2u_2(t)\ . \tag{2.2}$$

Finally, using the fact that

$$C_1\dot{x}_1(t) = i_1(t), \quad C_2\dot{x}_2(t) = i_2(t)\ ,$$

$$\implies x_3(t) = C_1\dot{x}_1(t) - C_2\dot{x}_2(t)\ ,$$

we can write

$$C_1\dot{x}_1(t) - C_2\dot{x}_2(t) = x_3(t)\ . \tag{2.3}$$

Packaging equations (2.1), (2.2) and (2.3) in matrix form and isolating the state-variable time derivatives gives

$$\begin{bmatrix} R_1C_1 & 0 & L \\ 0 & R_2C_2 & -L \\ C_1 & -C_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -R_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\ ,$$

and therefore

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} R_1C_1 & 0 & L \\ 0 & R_2C_2 & -L \\ C_1 & -C_2 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -R_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right)\ ,$$

where

$$\begin{bmatrix} R_1C_1 & 0 & L \\ 0 & R_2C_2 & -L \\ C_1 & -C_2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{C_1(R_1+R_2)} & \frac{1}{C_1(R_1+R_2)} & \frac{R_2}{C_1(R_1+R_2)} \\ \frac{1}{C_2(R_1+R_2)} & \frac{1}{C_2(R_1+R_2)} & \frac{-R_1}{C_2(R_1+R_2)} \\ \frac{R_2}{L(R_1+R_2)} & \frac{-R_1}{L(R_1+R_2)} & \frac{-R_1R_2}{L(R_1+R_2)} \end{bmatrix}$$

Multiplying through by the inverse yields the state equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1(R_1+R_2)} & -\frac{1}{C_1(R_1+R_2)} & \frac{R_2}{C_1(R_1+R_2)} \\ -\frac{1}{C_2(R_1+R_2)} & -\frac{1}{C_2(R_1+R_2)} & -\frac{R_1}{C_2(R_1+R_2)} \\ -\frac{R_2}{L(R_1+R_2)} & \frac{R_1}{L(R_1+R_2)} & -\frac{R_1R_2}{L(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1(R_1+R_2)} & -\frac{R_2}{C_1(R_1+R_2)} \\ \frac{1}{C_2(R_1+R_2)} & -\frac{R_2}{C_2(R_1+R_2)} \\ \frac{R_2}{L(R_1+R_2)} & \frac{R_1R_2}{L(R_1+R_2)} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\ ,$$

and the output equation:

$$y(t) = \begin{bmatrix} -\frac{R_2}{(R_1+R_2)} & \frac{R_1}{(R_1+R_2)} & -\frac{R_1R_2}{(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{R_2}{(R_1+R_2)} & \frac{R_1R_2}{(R_1+R_2)} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\ .$$

Question 3. *Consider the matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) *Compute the characteristic polynomial of A and find the eigenvalues of A.*

(b) *Compute three linearly independent eigenvectors for A.*

(c) *Find a similarity transformation L such that $\hat{A} = L^{-1}AL$ is a diagonal matrix.*

(d) *Compute e^{At} as a function of t.*

(e) *Compute $\sin(At)$ as a function of t.*

Solution 3.

(a) Compute the characteristic polynomial:

$$\det(sI - A) = \det \begin{bmatrix} s-2 & 0 & 0 \\ -1 & s-1 & 0 \\ 0 & 0 & s-3 \end{bmatrix}.$$

Expanding along the third row:

$$\det(sI - A) = (s-3) \times \det \begin{bmatrix} s-2 & 0 \\ -1 & s-1 \end{bmatrix}.$$

Compute the determinant of the 2×2 matrix:

$$\det \begin{bmatrix} s-2 & 0 \\ -1 & s-1 \end{bmatrix} = (s-2)(s-1) - 0 = s^2 - 3s + 2$$

Thus:

$$\det(sI - A) = (s-3)(s^2 - 3s + 2).$$

Factorizing:

$$\det(sI - A) = (s-3)(s-2)(s-1).$$

Hence, the eigenvalues are $\lambda = 3, 2, 1$.

(b) Solve $Av = \lambda v$ for each eigenvalue λ:
For $\lambda = 3$:

$$(A - 3I)v = 0 \implies \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0.$$

A solution is $v_1 = [0, 0, 1]^\top$.
For $\lambda = 2$:

$$(A - 2I)v = 0 \implies \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} v = 0.$$

A solution is $v_2 = [1, 1, 0]^\top$.
For $\lambda = 1$:

$$(A - I)v = 0 \implies \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} v = 0.$$

A solution is $v_3 = [0, 1, 0]^\top$.

(c) Let $M = [v_1, v_2, v_3]$ be the matrix of eigenvectors:

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that

$$AM = M\Lambda, \implies A = M\Lambda M^{-1},$$

hence, $L = M$ and $\hat{A} = \Lambda$ where

$$\hat{A} = \Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) Compute e^{At}:

$$e^{At} = Me^{\hat{A}t}M^{-1},$$

where

$$e^{\hat{A}t} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}.$$

To compute M^{-1}, first find the determinant of M:

$$\det(M) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

$$\det(M) = 0 - 1(-1) + 0 = 1.$$

Since $\det(M) = 1$, the cofactor matrix is:

$$\text{Cofactor matrix } = \text{cof}(M) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus the inverse matrix is:

$$M^{-1} = \frac{1}{\det(M)}\text{cof}(M)^T = \frac{1}{\det(M)}\text{adj}(M) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Now compute $Me^{\hat{A}t}$:

$$Me^{\hat{A}t} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} = \begin{bmatrix} 0 & e^{2t} & 0 \\ 0 & e^{2t} & e^{t} \\ e^{3t} & 0 & 0 \end{bmatrix}.$$

Finally:

$$e^{At} = Me^{\hat{A}t}M^{-1} = \begin{bmatrix} 0 & e^{2t} & 0 \\ 0 & e^{2t} & e^{t} \\ e^{3t} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 & 0 \\ e^{2t} - e^{t} & e^{t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

(e) Compute $\sin(At)$:

$$\sin(At) = M\sin(\hat{A}t)M^{-1},$$

where

$$\sin(\hat{A}t) = \begin{bmatrix} \sin(3t) & 0 & 0 \\ 0 & \sin(2t) & 0 \\ 0 & 0 & \sin(t) \end{bmatrix}.$$

Now compute $M\sin(\hat{A}t)$:

$$M\sin(\hat{A}t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sin(3t) & 0 & 0 \\ 0 & \sin(2t) & 0 \\ 0 & 0 & \sin(t) \end{bmatrix} = \begin{bmatrix} 0 & \sin(2t) & 0 \\ 0 & \sin(2t) & \sin(t) \\ \sin(3t) & 0 & 0 \end{bmatrix}$$

Finally:

$$e^{At} = M\sin(\hat{A}t)M^{-1} = \begin{bmatrix} 0 & \sin(2t) & 0 \\ 0 & \sin(2t) & \sin(t) \\ \sin(3t) & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \sin(2t) & 0 & 0 \\ \sin(2t) - \sin(t) & \sin(t) & 0 \\ 0 & 0 & \sin(3t) \end{bmatrix}.$$

Question 4. *Consider a linear continuous-time system described by the equations*

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + \alpha x_2 + u(t) \\ \dot{x}_2(t) &= x_1(t) + x_2(t) - \alpha x_2(t) \\ y(t) &= x_1(t) \end{aligned}$$

with $\alpha \in \mathbb{R}$ *and constant,* $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$ *and* $u(t) \in \mathbb{R}$.

(a) *Let* $u(t) = 0$*, for all* $t \geq 0$*. Compute the equilibrium points of the system as a function of* α.

(b) *Assume now* $u(t) = u(0)$*, for all* $t \geq 0$*, where* $u(t) \neq 0$*. Compute the equilibrium points of the system as a function of* α.

(c) *Discuss similarities and differences between the results in part (a) and part (b).*

Solution 4.

(a) To begin with, note that

$$A = \begin{bmatrix} 1 & \alpha \\ 1 & 1-\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and that $\det(A) = 1 - 2\alpha$. If $1 - 2\alpha \neq 0$, the matrix A is invertible. Hence the only equilibrium for $u(t) = u(0) = 0$ is $x(0) = -A^{-1}Bu(0) = 0$ for all $t \geq 0$. If $\alpha = 1/2$ and $\det(A) = 0$, then to find the equilibrium points, we need to solve the equations $\dot{x}_1(t) = \dot{x}_2(t) = 0$ for $u(t) = 0$, that is

$$0 = x_1(t) + \frac{1}{2}x_2(t) \qquad 0 = x_1(t) + x_2(t) - \frac{1}{2}x_2(t) = x_1(t) + \frac{1}{2}x_2(t)$$

This means that all equilibrium points described by

$$x(t) = x(0) = \delta \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

where δ can be any number. Thus, when $\alpha = 1/2$ and $u(t) = u(0) = 0$, the system has infinitely many equilibrium points on a straight line.

(b) As for part (a), if $1 - 2\alpha \neq 0$, the matrix A is invertible hence the only equilibrium for $u(t) = u(0) \neq 0$ is $x(0) = -A^{-1}Bu(0) = 0$ for all $t \geq 0$. That is

$$x(0) = \frac{u(0)}{1-2\alpha} \begin{bmatrix} 1-\alpha & -\alpha \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{u(0)}{1-2\alpha} \begin{bmatrix} 1-\alpha \\ -1 \end{bmatrix} \text{ or } \frac{u(0)}{2\alpha-1} \begin{bmatrix} \alpha-1 \\ 1 \end{bmatrix}$$

If $\alpha = 1/2$ and $\det(A) = 0$, then to find the equilibrium points, we need to solve the equations $\dot{x}_1(t) = \dot{x}_2(t) = 0$ for $u(t) = u(t) \neq 0$, that is

$$0 = x_1(t) + \frac{1}{2}x_2(t) + u(0) \qquad 0 = x_1(t) + \frac{1}{2}x_2(t)$$

These equations do not have any solution for $u(0) \neq 0$; that is, the system does not have any equilibrium points.

(c) If the matrix A is invertible, regardless of the value of the input signal, the system has one equilibrium point. If A is not invertible, the existence of equilibrium points depends upon the value of the input signal. If $u(t) = u(0) = 0$, there are infinitely many equilibria, whereas if $u(t) = u(0) \neq 0$, there are no equilibria.